Sharp Upper Bounds for the Orders of The Recurrences Outputted by the Zeilberger and q-Zeilberger Algorithms

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"Er muß sozusagen die Leiter wegwerfen, nachdem er auf ihr hinaufgestiegen ist"
——L. Wittgenstein (Logisch-philosophische Abhandlung, 6.54)

Abstract: We do what the title promises, and as a bonus, we get much simplified versions of these algorithms, that do not make any explicit mention of Gosper's algorithm.

Notation. For k integer, $(z)_k := z(z+1)\dots(z+k-1), [a]_k := (1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1})$, if $k \geq 0$ and $(z)_k := 1/(z+k)_{-k}, [a]_k := 1/[a+k]_{-k}$, if $k < 0$. For a *Laurent polynomial* $p(t)$ of t, $deg(p)$ is the *degree*, and $ldeg(p)$ is the *low-degree*, e.g., if $p = 4t^{-3} + 2t^{-1} + 4 + 3t + t^2$, $deg(p) = 2, ldeg(p) = -3$.

Theorem. Let

$$F(n,k) = POL(n,k) \cdot H(n,k) \quad , \qquad (ProperHypergeometric)$$

where $POL(n,k)$ is a polynomial in (n,k) and

$$H(n,k) = \frac{\prod_{j=1}^{A} (a''_j)_{a'_j n + a_j k} \prod_{j=1}^{B} (b''_j)_{b'_j n - b_j k}}{\prod_{j=1}^{C} (c''_j)_{c'_j n + c_j k} \prod_{j=1}^{D} (d''_j)_{d'_j n - d_j k}} z^k \quad , \qquad (PureHypergeometric)$$

where the $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are *non-negative integers*, and z, $a''_j, b''_j, c''_j, d''_j$ are *commuting indeterminates*. We also assume that the factorization in $(ProperHypergeometric)$ is *maximal*, i.e. $POL(n,k)$ is of the largest possible degree. Let

$$L = \max \left(\sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j \quad , \quad \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j \right) \quad . \qquad (ZBound)$$

There exist polynomials in n, $e_0(n), \dots, e_L(n)$, *not all zero*, and a rational function $R(n,k)$ such that $G(n,k) := R(n,k)F(n,k)$ satisfies

$$\sum_{i=0}^{L} e_i(n) F(n+i,k) = G(n,k+1) - G(n,k) \quad . \qquad (Zpair)$$

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Furthermore, *in general*, L cannot be made any smaller.

Proof: Let

$$\overline{H}(n,k) = \frac{\prod_{j=1}^{A} (a''_j)_{a'_j n + a_j k} \prod_{j=1}^{B} (b''_j)_{b'_j n - b_j k}}{\prod_{j=1}^{C} (c''_j)_{c'_j (n+L) + c_j k} \prod_{j=1}^{D} (d''_j)_{d'_j (n+L) - d_j k}} z^k \quad ,$$

$$f(k) = z \prod_{j=1}^{A} (a'_j n + a_j k + a''_j)_{a_j} \prod_{j=1}^{D} (d'_j (n+L) - d_j k + d''_j - d_j)_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^{B} (b'_j n - b_j k + b''_j - b_j)_{b_j} \prod_{j=1}^{C} (c'_j (n+L) + c_j k + c''_j)_{c_j} \quad .$$

Note that $\overline{H}(n,k+1)/\overline{H}(n,k) = f(k)/g(k)$. Write

$$G(n,k) = g(k-1) X(k) \overline{H}(n,k) \quad . \qquad (Ansatz)$$

Substituting into $(Zpair)$ and dividing both sides by $\overline{H}(n,k)$, shows that it is equivalent to

$$f(k) X(k+1) - g(k-1) X(k) - h(k) = 0 \quad , \qquad (Gosper)$$

where

$$h(k) := \sum_{i=0}^{L} e_i(n) POL(n+i,k) \cdot \frac{H(n+i,k)}{\overline{H}(n,k)} \quad .$$

Note that $h(k)$ is a *polynomial* since

$$\frac{H(n+i,k)}{\overline{H}(n,k)} =$$

$$\prod_{j=1}^{A} (a'_j n + a_j k + a''_j)_{i a'_j} \prod_{j=1}^{B} (b'_j n - b_j k + b''_j)_{i b'_j} \prod_{j=1}^{C} (c'_j n + c_j k + c''_j + i c'_j)_{(L-i) c'_j} \prod_{j=1}^{D} (d'_j n - d_j k + d''_j + i d'_j)_{(L-i) d'_j} \quad .$$

We claim that $(Gosper)$ can always be solved (non-trivially) with $X(k)$ being a polynomial of degree $M := deg(h) - max(deg(f), deg(g))$. Writing

$$X(k) = \sum_{i=0}^{M} x_i(n) k^i \quad , \qquad (Ansatz1)$$

substituting into $(Gosper)$, and setting all the coefficients to 0, yields $deg(h)+1$ *homogeneous* linear equations for the $M+L+2$ unknowns $e_0(n), \ldots, e_L(n)$, and $x_0(n), \ldots, x_M(n)$. For such a *not-all-zero* solution to exist, we need $\#\,unknowns\, - \,\#\,equations\, - 1 \geq 0$, i.e. $(M+L+2) - (deg(h)+1) - 1 \geq 0$, i.e. $L \geq max(deg(f), deg(g))$. But

$$deg(f) = \sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j \quad , \quad deg(g) = \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \ldots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \ldots, e_L(n)$, *and* $x_0(n), \ldots, x_M(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(k)$ is zero and $(Gosper)$ becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a polynomial, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(n, k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a''_j, b''_j, c''_j, d''_j$ arbitrary (commuting) *symbols*.

To prove that $(Zbound)$ is sharp, take $F(n, k) = 1/((1)_k (1)_{n-k})$ and note that L cannot be 0, since otherwise it would have been gosperable with respect to k, but it is not, as can be seen by performing the Gosper algorithm[G] on it. □

q-Theorem. Let

$$F(n, k) = POL(q^n, q^k) \cdot H(n, k) \quad , \qquad (qProperHypergeometric)$$

where $POL(q^n, q^k)$ is a Laurent polynomial in (q^n, q^k), and

$$H(n, k) = \frac{\prod_{j=1}^{A} [a''_j]_{a'_j n + a_j k} \prod_{j=1}^{B} [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^{C} [c''_j]_{c'_j n + c_j k} \prod_{j=1}^{D} [d''_j]_{d'_j n - d_j k}} q^{Jk(k-1)/2} z^k \quad , \qquad (qPureHypergeometric)$$

where the $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are *non-negative integers*, and z, $a''_j, b''_j, c''_j, d''_j$ are *indeterminates*, and J is an integer. We also assume that the factorization in $(qProperHypergeometric)$ is *maximal*, i.e. $POL(q^n, q^k)$ is as 'large' as possible. Let

$$L = \max\left(J + \sum_{j=1}^{A} {a_j}^2 \; , \; \sum_{j=1}^{C} {c_j}^2 \right) + \max\left(-J + \sum_{j=1}^{D} {d_j}^2 \; , \; \sum_{j=1}^{B} {b_j}^2 \right) \quad . \qquad (qZBound)$$

There exist polynomials in q^n, $e_0(q^n), \ldots, e_L(q^n)$, *not all zero*, and a rational function $R(q^n, q^k)$ such that $G(n, k) := R(q^n, q^k) F(n, k)$ satisfies

$$\sum_{i=0}^{L} e_i(q^n) F(n+i, k) = G(n, k+1) - G(n, k) \quad . \qquad (qZpair)$$

Furthermore, *in general*, L, cannot be made any smaller.

Proof: Let

$$\overline{H}(n, k) = \frac{\prod_{j=1}^{A} [a''_j]_{a'_j n + a_j k} \prod_{j=1}^{B} [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^{C} [c''_j]_{c'_j (n+L) + c_j k} \prod_{j=1}^{D} [d''_j]_{d'_j (n+L) - d_j k}} q^{Jk(k-1)/2} z^k \quad ,$$

$$f(k) = zq^{Jk} \prod_{j=1}^{A} [a'_j n + a_j k + a''_j]_{a_j} \prod_{j=1}^{D} [d'_j (n+L) - d_j k + d''_j - d_j]_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^{B} [b'_j n - b_j k + b''_j - b_j]_{b_j} \prod_{j=1}^{C} [c'_j (n+L) + c_j k + c''_j]_{c_j} \quad .$$

Note that $\overline{H}(n, k+1)/\overline{H}(n,k) = f(k)/g(k)$. Write

$$G(n,k) = g(k-1) X(k) \overline{H}(n,k) \quad . \qquad (qAnsatz)$$

Substituting into $(Zpair)$ and dividing both sides by $\overline{H}(n,k)$, shows that it is equivalent to

$$f(k)X(k+1) - g(k-1)X(k) - h(q^k) = 0 \quad , \qquad (qGosper)$$

where

$$h(q^k) := \sum_{i=0}^{L} e_i(q^n) POL(q^n q^i, q^k) \cdot \frac{H(n+i,k)}{\overline{H}(n,k)} \quad .$$

Note that $h(q^k)$ is a *Laurent polynomial* (in q^k) since

$$\frac{H(n+i,k)}{\overline{H}(n,k)} =$$

$$\prod_{j=1}^{A} [a'_j n + a_j k + a''_j]_{i a'_j} \prod_{j=1}^{B} [b'_j n - b_j k + b''_j]_{i b'_j} \prod_{j=1}^{C} [c'_j n + c_j k + c''_j + i c'_j]_{(L-i)c'_j} \prod_{j=1}^{D} [d'_j n - d_j k + d''_j + i d'_j]_{(L-i)d'_j} \quad .$$

Let

$$M_1 := -ldeg(h) - max(-ldeg(f), -ldeg(g)) \quad , \quad M_2 := deg(h) - max(deg(f), deg(g)) \quad .$$

We claim that $(qGosper)$ can always be solved (non-trivially) with $X(k)$ a Laurent polynomial of q^k of low-degree $-M_1$ and degree M_2. Writing

$$X(k) = \sum_{i=-M_1}^{M_2} x_i(q^n)(q^k)^i \quad , \qquad (qAnsatz1)$$

substituting into $(qGosper)$, and setting all the coefficients to 0, yields $-ldeg(h)+deg(h)+1$ *homogeneous* linear equations for the M_1+M_2+L+2 unknowns $e_0(q^n), \dots, e_L(q^n)$, and $x_{-M_1}(q^n), \dots, x_{M_2}(q^n)$. For such a *not-all-zero* solution to exist, we need $\#\,unknowns\ -\ \#\,equations\ -1 \geq 0$, i.e. $(M_1 + M_2 + L + 2) - (-ldeg(h) + deg(h) + 1) - 1 \geq 0$, i.e. $L \geq max(deg(f), deg(g)) + max(-ldeg(f), -ldeg(g))$. But

$$deg(f) = J + \sum_{j=1}^{A} {a_j}^2 \quad , \quad -ldeg(f) = -J + \sum_{j=1}^{D} {d_j}^2 \quad , \quad deg(g) = \sum_{j=1}^{C} {c_j}^2 \quad , \quad -ldeg(g) = \sum_{j=1}^{B} {b_j}^2 \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(q^n), \ldots, e_L(q^n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(q^n), \ldots, e_L(q^n)$, *and* $x_{-M_1}(q^n), \ldots, x_{M_2}(q^n)$ to be zero). But if all the $e_i(q^n)$'s are zero, then $h(q^k)$ is zero and (*qGosper*) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a Laurent polynomial in q^k, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(q^n, q^k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a''_j, b''_j, c''_j, d''_j$ arbitrary *symbols.*

To prove that (*qZbound*) is sharp, take $F(n,k) = q^{k(k-1)/2}/([1]_k[1]_{n-k})$, and note that L cannot be 0, since otherwise it would have been q-gosperable with respect to k, but it is not, as can be seen by performing the q-Gosper algorithm ([Kor][PR], or use `qEKHAD`) on it. □

Comments

1. The bounds in (*ZBound*) and (*qZBound*) considerably improve those of [WZ] (Theorems 3.1 and 5.1, see also [PWZ] and [Koe]), that relied on Sister Celine's Technique, and, as we proved, are *sharp* for the *generic* case. However, sometimes a system of linear equations with more equations than unknowns *does* have a non-trivial solution, and also, sometimes one can find higher-degree polynomial solutions to (*Gosper*) and (*qGosper*), so in *specific* cases, it is possible to have recurrences of lower order. This is the case for all the non-trivial classical hypergeometric (binomial-coefficient) sums that admit a closed-form evaluation.

2. The proofs imply new, simplified, versions of the Zeilberger[Z1][Z2][PS] and q-Zeilberger[Kor][PR] algorithms. These new versions do not rely on Gosper's algorithm *explicitly*, but, of course, were inspired by it. In fact, they were designed by applying the Zeilberger and q-Zeilberger algorithms *once and for all*, to the *generic* cases. It so happens, that in this case, a simplified version of Gosper (and q-Gosper) suffices, and it is so simple that it can be incorporated implicitly. So old-Zeilberger (and hence Gosper and q-Gosper) is the Wittgensteinian ladder that we *must* throw away after we climbed it.

The *running-time* complexity of these new versions are comparable to the old versions, but their *program-length complexity* (in the sense of Chaitin-Kolmogorov) are considerably smaller.

The simplified Zeilberger and q-Zeilberger algorithms, apply also to *specific*, non-generic summands. Start by taking $L = 0$ and try the *ansatzes* (*Ansatz*) and (*qAnsatz*), but with M (for the q-case: M_1, M_2) possibly larger than the ones in the theorem (which are determined by plugging them into (*Gosper*) or (*qGosper*), and equating the leading coefficient(s), and finding out whether they can vanish for integral M (or M_1, M_2)). Then one solves the resulting set of linear equations. If there is no non-trivial solution, then one increases L by 1, until success is reached. The theorems guarantee that eventually we will succeed, at worst, with the L's given by (*ZBound*) and (*qZBound*).

3. These simplified versions are implemented in the Maple packages `ZEILBERGER` and `qZEILBERGER` available from
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ sharpZ.html .

4. The present article was intentionally written in a terse, unmotivated, style, in order to emphasize its simplicity and self-containedness. Readers who wish to see more motivation are welcome to look at an earlier version, that only covers the ordinary case, that is also available from the above webpage.

5. Sometimes the original Zeilberger algorithms work even when the summand $F(n,k)$ is *not* proper-hypergeometric, see [A] and [CHM]. Hence the new simplified versions do not completely supersede the old versions.

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