Polynomial Reduction and Super Congruences

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Abstract

Based on a reduction processing, we rewrite a hypergeometric term as the sum of the difference of a hypergeometric term and a reduced hypergeometric term (the reduced part, in short). We show that when the initial hypergeometric term has a certain kind of symmetry, the reduced part contains only odd or even powers. As applications, we derived two infinite families of super-congruences.

1 Introduction

In recent years, many super congruences involving combinatorial sequences are discovered, see for example, Sun [16]. The standard methods for proving these congruences include combinatorial identities [18], Gauss sums [5], symbolic computation [14] et al.

We are interested in the following super congruence conjectured by van

Hamme [19]

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(4k+1)\left(\frac{(1/2)_k}{(1)_k}\right)^3 \equiv (-1)^{\frac{p-1}{2}}p \pmod{p^3},$$

where p is a odd prime and $(a)_k = a(a+1)\cdots(a+k-1)$ is the rising factorial. This congruence was proved by Mortenson [13] Zudilin [21] and Long [12] by different methods. Sun [17] proved a stronger version for prime $p \geq 5$

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(4k+1)\left(\frac{(1/2)_k}{(1)_k}\right)^3 \equiv (-1)^{\frac{p-1}{2}}p + p^3E_{p-3} \pmod{p^4},$$

where E_n is the n-th Euler number defined by

$$\frac{2}{e^x+e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

A similar congruence was given by van Hamme [19] for $p \equiv 1 \pmod 4$:

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\left(\frac{(1/2)_k}{(1)_k}\right)^4 \equiv p \pmod{p^3}.$$

Long [12] showed that in fact the above congruence holds for arbitrary odd prime modulo p^4. Motivated by these two congruences, Guo [8] proposed the following conjectures (corrected version).

Conjecture 1.1 • *For any odd prime p, positive integer r and odd integer m, there exists an integer $a_{m.p}$ such that*

$$\sum_{k=0}^{\frac{p^r-1}{2}}(-1)^k(4k+1)^m\left(\frac{(1/2)_k}{(1)_k}\right)^3 \equiv a_{m,p}p^r(-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}. \tag{1.1}$$

• *For any odd prime $p > (m-1)/2$, positive integer r and odd integer m, there exists an integer $b_{m,p}$ such that*

$$\sum_{k=0}^{\frac{p^r-1}{2}}(4k+1)^m\left(\frac{(1/2)_k}{(1)_k}\right)^4 \equiv b_{m,p}p^r \pmod{p^{r+3}}. \tag{1.2}$$

Liu [11] and Wang [20] confirmed the conjectures for $r = 1$ and some initial values m. Jana and Kalita [10] and Guo [9] confirmed (1.1) for $m = 3$ and $r \geq 1$. We will prove a stronger version of (1.1) for the case of $r = 1$ and arbitrary odd m and a weaker version of (1.2) for the case of $r = 1$ and arbitrary odd m by a reduction process.

Recall that a hypergeometric term t_k is a function of k such that t_{k+1}/t_k is a rational function of k. Our basic idea is to rewrite the product of a polynomial $f(k)$ in k and a hypergeometric term t_k as

$$f(k)t_k = \Delta(g(k)t_k) + h(k)t_k = (g(k+1)t_{k+1} - g(k)t_k) + h(k)t_k,$$

where $g(k), h(k)$ are polynomials in k such that the degree of $h(k)$ is bounded. To this aim, we construct $x(k)$ such that $\Delta x(k)t_k$ equals the product of t_k and a polynomial $u(k)$ and that $f(k)$ and $u(k)$ has the same leading term. Then we have

$$f(k)t_k - \Delta x(k)t_k = (f(k) - u(k))t_k$$

is the product of t_k and a polynomial of degree less than $f(k)$. We call such a reduction process one reduction step. Continuing this reduction process, we finally obtain a polynomial $h(k)$ with bounded degree. We will show that for $t_k = \left(\frac{(1/2)_k}{(1)_k}\right)^r$, $r = 3, 4$ and an arbitrary polynomial of form $(4k+1)^m$ with m odd, the reduced polynomial $h(k)$ can be taken as $(4k+1)$. This enables us to reduce the congruences (1.1) and (1.2) to the special case of $m = 1$, which is known for $r = 1$.

We notice that Pirastu-Strehl [15] and Abramov [1, 2] gave the minimal decomposition when t_k is a rational function, Abramov-Petkovšek [3, 4] gave the minimal decomposition when t_k is a hypergeometric term, and Chen-Huang-Kauers-Li [6] applied the reduction to give an efficient creative telescoping algorithm. These algorithms concern a general hypergeometric term. While we focus on a kind of special hypergeometric term so that the reduced part $h(k)t_k$ has a nice form.

The paper is organized as follows. In Section 2, we consider the reduction process for a general hypergeometric term t_k. Then in Section 3 we consider those t_k with the property $a(k)$ is a shift of $-b(k)$, where $t_{k+1}/t_k = a(k)/b(k)$. As an application, we prove a stronger version of (1.1) for the case $r = 1$. Finally, we consider the case of $a(k)$ is a shift of $b(k)$, which corresponds to (1.2). In this case, we show that there is a rational number b_m instead of an integer such that (1.2) holds when $r = 1$.

2 The Difference Space and Polynomial Reduction

Let K be a field and $K[k]$ be the ring of polynomials in k with coefficients in K. Let t_k be a hypergeometric term. Suppose that

$$\frac{t_{k+1}}{t_k} = \frac{a(k)}{b(k)},$$

where $a(k), b(k) \in K[k]$. It is straightforward to verify that

$$\Delta_k \left(b(k-1)x(k)t_k\right) = (a(k)x(k+1) - b(k-1)x(k))t_k. \tag{2.1}$$

We thus define the *difference space* corresponding to $a(k)$ and $b(k)$ to be

$$S_{a,b} = \{a(k)x(k+1) - b(k-1)x(k) \colon x(k) \in K[k]\}.$$

We see that for $f(k) \in S_{a,b}$, we have $f(k)t_k = \Delta_k(p(k)t_k)$ for a certain polynomial $p(k) \in K[k]$.

Let $\mathbb{N}, \mathbb{Z}$ denote the set of nonnegative integers and the set of integers, respectively. Given $a(k), b(k) \in K[k]$, we denote

$$u(k) = a(k) - b(k-1), \tag{2.2}$$

$$d = \max\{\deg u(k), \deg a(k) - 1\}, \tag{2.3}$$

and

$$m_0 = -\operatorname{lc} u(k)/\operatorname{lc} a(k), \tag{2.4}$$

where $\operatorname{lc} p(k)$ denotes the leading coefficient of $p(k)$.

We first introduce the concept of degeneration.

Definition 2.1 *Let $a(k), b(k) \in K[k]$ and $u(k), m_0$ be given by* (2.2) *and* (2.4)*. If*

$$\deg u(k) = \deg a(k) - 1 \quad \textit{and} \quad m_0 \in \mathbb{N},$$

we say that the pair $(a(k), b(k))$ is degenerated*.*

We will see that the degeneration is closely related to the degrees of the elements in $S_{a,b}$.

Lemma 2.2 *Let $a(k), b(k) \in K[k]$ and d, m_0 be given by* (2.3) *and* (2.4). *For any polynomial $x(k) \in K[k]$, let*

$$p(k) = a(k)x(k+1) - b(k-1)x(k).$$

If $(a(k), b(k))$ is degenerated and $\deg x(k) = m_0$, then $\deg p(k) < d + m_0$; Otherwise, $\deg p(k) = d + \deg x(k)$.

Proof. Notice that

$$p(k) = u(k)x(k) + a(k)(x(k+1) - x(k)).$$

If the leading terms of $u(k)x(k)$ and $a(k)(x(k+1)-x(k))$ do not cancel, the degree of $p(k)$ is $d + \deg x(k)$. Otherwise, we have $\deg u(k) = \deg a(k) - 1$ and

$$\operatorname{lc} u(k) + \operatorname{lc} a(k) \cdot \deg x(k) = 0,$$

i.e., $\deg x(k) = m_0$. ∎

It is clear that $S_{a,b}$ is a subspace of $K[k]$, but is not a sub-ring of $K[k]$ in general. Let $[p(k)] = p(k) + S_{a,b}$ denote the coset of a polynomial $p(k)$. We see that the quotient space $K[k]/S_{a,b}$ is finite dimensional.

Theorem 2.3 *Let $a(k), b(k) \in K[k]$ and d, m_0 be given by* (2.3) *and* (2.4). *We have*

$$K[k]/S_{a,b} = \begin{cases} \langle [k^0], [k^1], \ldots, [k^{d-1}], [k^{d+m_0}] \rangle, & \textit{if } (a(k), b(k)) \textit{ is degenerated,} \\ \langle [k^0], [k^1], \ldots, [k^{d-1}] \rangle, & \textit{otherwise.} \end{cases}$$

Proof. For any nonnegative integer s, let

$$p_s(k) = a(k)(k+1)^s - b(k-1)k^s.$$

We first consider the case when the pair $(a(k), b(k))$ is not degenerated. By Lemma 2.2, we have

$$\deg p_s(k) = d + s, \quad \forall\, s \geq 0.$$

Suppose that $p(k)$ is a polynomial of degree $m \geq d$. Then

$$p'(k) = p(k) - \frac{\operatorname{lc} p(k)}{\operatorname{lc} p_{m-d}(k)} p_{m-d}(k) \tag{2.5}$$

is a polynomial of degree less than m and $p(k) \in [p'(k)]$. By induction on m, we derive that for any polynomial $p(k)$ of degree $\geq d$, there exists a polynomial $\tilde{p}(k)$ of degree $< d$ such that $p(k) \in [\tilde{p}(k)]$. Therefore,

$$K[k]/S_{a,b} = \langle [k^0], [k^1], \ldots, [k^{d-1}] \rangle.$$

Now assume that $(a(k), b(k))$ is degenerated. By Lemma 2.2,

$$\deg p_s(k) = d + s, \quad \forall\, s \neq m_0 \quad \text{and} \quad \deg p_{m_0}(k) < d + m_0.$$

The above reduction process (2.5) works well except for the polynomials $p(k)$ of degree $d + m_0$. But in this case,

$$p(k) - \operatorname{lc} p(k) \cdot k^{d+m_0}$$

is a polynomial of degree less than $d + m_0$. Then the reduction process continues until the degree is less than d. We thus derive that

$$K[k]/S_{a,b} = \langle [k^0], [k^1], \ldots, [k^{d-1}], [k^{d+m_0}] \rangle,$$

completing the proof. ∎

Example 2.1 *Let n be a positive integer and*

$$t_k = (-n)_k / k!,$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ is the raising factorial. Then

$$a(k) = k - n, \quad b(k) = k + 1,$$

and

$$S_{a,b} = \{(k-n) \cdot x(k+1) - k \cdot x(k) \colon x(k) \in K[k]\}.$$

We have

$$K[k]/S_{a,b} = \langle [k^n] \rangle$$

is of dimension one.

3 The case when $a(k) = -b(k+\alpha)$

In this section, we consider the case when $a(k) = -b(k+\alpha)$ and $b(k)$ has a symmetric property. We will show that in this case, the reduction process maintains the symmetric property. Notice that in this case

$$u(k) = a(k) - b(k-1) = -b(k+\alpha) - b(k-1)$$

has the same degree as $a(k)$, the pair $(a(k), b(k))$ is not degenerated.

We first consider the relation between the symmetric property and the expansion of a polynomial.

Lemma 3.1 *Let $p(k) \in K[k]$ and $\beta \in K$. Then the following two statements are equivalent.*

(1) $p(\beta + k) = p(\beta - k)$ *($p(\beta + k) = -p(\beta - k)$, respectively).*

(2) *$p(k)$ is the linear combination of $(k-\beta)^{2i}$, $i = 0, 1, \ldots$ ($(k-\beta)^{2i+1}$, $i = 0, 1, \ldots$, respectively).*

Proof. Suppose that

$$p(\beta + k) = \sum_i c_i k^i.$$

Then

$$p(\beta - k) = \sum_i c_i (-k)^i.$$

Therefore,

$$p(\beta + k) = p(\beta - k) \Longleftrightarrow c_{2i+1} = 0,\ i = 0, 1, \ldots.$$

The case of $p(\beta + k) = -p(\beta - k)$ can be proved in a similar way. ∎

Now we are ready to state the main theorem.

Theorem 3.2 *Let $a(k), b(k) \in K[k]$ such that*

$$a(k) = -b(k+\alpha) \quad \textit{and} \quad b(\beta + k) = \pm b(\beta - k),$$

for some $\alpha, \beta \in K$. Then for any non-negative integer m, we have

$$[(k+\gamma)^{2m}] \in \left\langle [(k+\gamma)^{2i}] \colon 0 \le 2i < \deg a(k) \right\rangle$$

and

$$[(k+\gamma)^{2m+1}] \in \left\langle [(k+\gamma)^{2i+1}]\colon 0 \le 2i+1 < \deg a(k) \right\rangle,$$

where

$$\gamma = -\beta + \frac{\alpha-1}{2}. \tag{3.1}$$

Proof. We only prove the case of $b(\beta+k) = b(\beta-k)$. The case of $b(\beta+k) = -b(\beta-k)$ can be proved in a similar way. By Lemma 3.1, we may assume that

$$b(k) = b_r(k-\beta)^r + b_{r-2}(k-\beta)^{r-2} + \cdots + b_0,$$

where $r = \deg a(k) = \deg b(k)$ is even and $b_r, b_{r-2}, \ldots, b_0 \in K$ are the coefficients.

Since $(a(k), b(k))$ is not degenerated, taking

$$x(k) = x_s(k) = -\frac{1}{2}\left(k+\gamma-\frac{1}{2}\right)^s \tag{3.2}$$

in Lemma 2.2, we derive that

$$p_s(k) = a(k)x_s(k+1) - b(k-1)x_s(k) \tag{3.3}$$

is a polynomial of degree $s+r$. More explicitly, we have

$$p_s(k) = \frac{1}{2}\left(b(k+\alpha)\left(k+\gamma+\frac{1}{2}\right)^s + b(k-1)\left(k+\gamma-\frac{1}{2}\right)^s\right)$$

is a polynomial with leading term $b_r k^{s+r}$.

Notice that

$$p_s(-\gamma+k) = \frac{1}{2}\left(b(k+\alpha-\gamma)\left(k+\frac{1}{2}\right)^s + b(k-\gamma-1)\left(k-\frac{1}{2}\right)^s\right)$$

and

$$\begin{aligned} p_s(-\gamma-k) &= \frac{1}{2}\left(b(-k+\alpha-\gamma)\left(-k+\frac{1}{2}\right)^s + b(-k-\gamma-1)\left(-k-\frac{1}{2}\right)^s\right) \\ &= \frac{(-1)^s}{2}\left(b(-k+\alpha-\gamma)\left(k-\frac{1}{2}\right)^s + b(-k-\gamma-1)\left(k+\frac{1}{2}\right)^s\right). \end{aligned}$$

Since $b(\beta+k) = b(\beta-k)$, i.e., $b(k) = b(2\beta-k)$, we deduce that

$$\begin{aligned} &p_s(-\gamma-k) \\ &= \frac{(-1)^s}{2}\left(b(2\beta+k-\alpha+\gamma)\left(k-\frac{1}{2}\right)^s + b(2\beta+k+\gamma+1)\left(k+\frac{1}{2}\right)^s\right). \end{aligned}$$

By the relation (3.1), we derive that

$$p_s(-\gamma - k) = (-1)^s p_s(-\gamma + k).$$

Suppose that $p(k)$ is a linear combination of the even powers of $(k+\gamma)$ and $\deg p(k) \geq r$. By Lemma 3.1, we have $p(-\gamma-k) = p(-\gamma+k)$ and thus

$$p'(k) = p(k) - \frac{\operatorname{lc} p(k)}{b_r} \cdot p_{\deg p(k) - r}(k)$$

also satisfies $p'(-\gamma-k) = p'(-\gamma+k)$ since $\deg p(k)$ and r are both even. It is clear that $p(k) \in [p'(k)]$ and the degree of $p'(k)$ is less than the degree of $p(k)$. Continuing this reduction process, we finally derive that $p(k) \in [\tilde{p}(k)]$ for some polynomial $\tilde{p}(k)$ with degree $< r$ and satisfying $\tilde{p}(-\gamma-k) = \tilde{p}(-\gamma+k)$. Therefore,

$$[p(k)] \in \langle [(k+\gamma)^{2i}] \colon 0 \leq 2i < r \rangle.$$

Suppose that $p(k)$ is a linear combination of the odd powers of $(k+\gamma)$ and $\deg p(k) \geq r$. Then we have $p(-\gamma-k) = -p(-\gamma+k)$ and thus

$$p'(k) = p(k) - \frac{\operatorname{lc} p(k)}{b_r} \cdot p_{\deg p(k) - r}(k)$$

also satisfies $p'(-\gamma-k) = -p'(-\gamma+k)$. Continuing this reduction process, we finally derive that

$$[p(k)] \in \langle [(k+\gamma)^{2i+1}] \colon 0 \leq 2i+1 < r \rangle.$$

This completes the proof. ∎

We may further require to express $[(k+\gamma)^m]$ as an integral linear combination of $[(k+\gamma)^i]$, $0 \leq i < r$ when $b(k) = (k+1)^r$.

Theorem 3.3 *Let*

$$t_k = (-1)^k \left(\frac{(\alpha)_k}{k!} \right)^r,$$

where r is a positive integer and α is a rational number with denominator D. Then for any positive integer m, there exist integers $a_0, \ldots, a_{r-1}$ and a polynomial $x(k) \in \mathbb{Z}[k]$ such that

$$(2Dk + D\alpha)^m t_k = \sum_{i=0}^{r-1} a_i (2Dk + D\alpha)^i t_k + \Delta_k \left(2^{r-1} (Dk)^r x(2Dk) t_k \right).$$

Moreover, $a_i = 0$ if $i \not\equiv m \pmod 2$.

Proof. We have
$$\frac{t_{k+1}}{t_k} = \frac{-(k+\alpha)^r}{(k+1)^r}.$$
Let
$$a(k) = -(k+\alpha)^r \quad \text{and} \quad b(k) = (k+1)^r.$$
We see that it is the case of $\beta = -1$ and $\gamma = \alpha/2$ of Theorem 3.2. From (2.1), we derive that
$$\Delta_k(k^r x_s(k)t_k) = p_s(k)t_k, \tag{3.4}$$
where $x_s(k)$ and $p_s(k)$ are given by (3.2) and (3.3) respectively. Multiplying $(2D)^{s+r}$ on both sides, we obtain
$$\Delta_k(2^{r-1}(Dk)^r \tilde{x}_s(2Dk)t_k) = \tilde{p}_s(k')t_k, \tag{3.5}$$
where $k' = 2Dk + D\alpha$,
$$\tilde{x}_s(k) = -(k + D\alpha - D)^s, \tag{3.6}$$
and
$$\tilde{p}_s(k) = \frac{1}{2}\left((k+D\alpha)^r (k+D)^s + (k-D\alpha)^r (k-D)^s\right). \tag{3.7}$$
Notice that $\tilde{x}_s(k), \tilde{p}_s(k) \in \mathbb{Z}[k]$ and $\tilde{p}_s(k)$ is a monic polynomial of degree $s+r$. Moreover, $\tilde{p}_s(k)$ contains only even powers of k or only odd powers of k. Using $\tilde{p}_s(k)$ to do the reduction (2.5), we derive that there exists integers $c_m, c_{m-2}, \ldots$ such that
$$p(k) = k^m - c_m\tilde{p}_{m-r}(k) - c_{m-2}\tilde{p}_{m-r-2}(k) - \cdots$$
becomes a polynomial of degree less than r. Clearly, $p(k) \in \mathbb{Z}[k]$. Replacing k by k' and multiplying t_k, we derive that
$$(k')^m t_k = p(k')t_k + \Delta_k(2^{r-1}(Dk)^r(c_m\tilde{x}_{m-r}(2Dk) + c_{m-2}\tilde{x}_{m-r-2}(2Dk) + \cdots)t_k),$$
completing the proof. ∎

As an application, we confirm Conjecture 6 of [11].

Theorem 3.4 *Let*
$$S_m = \sum_{k=0}^{\frac{p-1}{2}} (-1)^k (4k+1)^m \left(\frac{(1/2)_k}{(1)_k}\right)^3.$$

For any positive odd integer m, there exist integers a_m and c_m such that

$$S_m \equiv a_m \left(p(-1)^{\frac{p-1}{2}} + p^3 E_{p-3} \right) + p^3 c_m \pmod{p^4}$$

holds for any prime $p \geq 5$.

Proof. Taking $r = 3$ and $\alpha = 1/2$ in Theorem 3.3, there exist an integer a_m and a polynomial $q_m(k) \in \mathbb{Z}[k]$ such that

$$(4k+1)^m t_k - a_m(4k+1)t_k = \Delta_k(32k^3 q_m(4k)t_k),$$

where $t_k = (-1)^k (\frac{1}{2})_k^3/(1)_k^3$. Summing over k from 0 to $\frac{p-1}{2}$, we derive that

$$S_m - a_m S_1 = 32\omega^3 q_m(4\omega)(-1)^\omega \left(\frac{(1/2)_\omega}{(1)_\omega} \right)^3,$$

where $\omega = \frac{p+1}{2}$. Noting that

$$\frac{(1/2)_\omega}{(1)_\omega} = p \frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{2i-1}{2i}$$

and

$$\frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{2i-1}{2i} = \frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{p-2i}{2i} \equiv (-1)^{\frac{p-1}{2}} \pmod{p},$$

we have

$$\left(\frac{(1/2)_\omega}{(1)_\omega} \right)^3 \equiv p^3 (-1)^{\frac{p-1}{2}} \pmod{p^4}.$$

Hence

$$S_m - a_m S_1 \equiv -32 p^3 \omega^3 q_m(4\omega) \pmod{p^4}$$

Let $c_m = -4q_m(2)$. We then have

$$S_m \equiv a_m S_1 + p^3 c_m \pmod{p^4}.$$

Sun [17] proved that for any prime $p \geq 5$,

$$S_1 \equiv (-1)^{\frac{p-1}{2}} p + p^3 E_{p-3} \pmod{p^4}.$$

Therefore,

$$S_m \equiv a_m \left(p(-1)^{\frac{p-1}{2}} + p^3 E_{p-3} \right) + p^3 c_m \pmod{p^4}.$$

∎

Remark 1. The coefficient a_m and the polynomial $q_m(k)$ can be computed by the extended Zeilberger's algorithm [7].

4 The case when $a(k) = b(k+\alpha)$

We first give a criterion on the degeneration of $(a(k), b(k))$.

Lemma 4.1 *Let $a(k), b(k) \in K[k]$ such that $a(k) = b(k+\alpha)$. Suppose that $-(\alpha+1)\deg a(k) \notin \mathbb{N}$. Then $(a(k), b(k))$ is not degenerated.*

Proof. Let $r = \deg a(k) = \deg b(k)$ and

$$u(k) = a(k) - b(k-1) = b(k+\alpha) - b(k-1).$$

It is clear that the coefficient of k^r in $u(k)$ is 0 and the coefficient of k^{r-1} in $u(k)$ is $\operatorname{lc} b(k) \cdot (\alpha+1)r$. Since $(\alpha+1)r \neq 0$, we derive that $\deg u(k) = r-1$. Thus,

$$-\operatorname{lc} u(k)/\operatorname{lc} a(k) = -\operatorname{lc} u(k)/\operatorname{lc} b(k) = -(\alpha+1)r.$$

Since $-(\alpha+1)r \notin \mathbb{N}$, the pair $(a(k), b(k))$ is not degenerated. ∎

When $a(k)$ is a shift of $b(k)$, we have a result similar to Theorem 3.2.

Theorem 4.2 *Let $a(k), b(k) \in K[k]$ such that*

$$a(k) = b(k+\alpha) \quad \textit{and} \quad b(\beta+k) = \pm b(\beta-k),$$

for some $\alpha, \beta \in K$. Assume further that $-(\alpha+1)\deg a(k) \notin \mathbb{N}$. Then for any non-negative integer m, we have

$$(k+\gamma)^{2m} \in \left\langle [(k+\gamma)^{2i}] \colon 0 \le 2i < \deg a(k) - 1 \right\rangle$$

and

$$(k+\gamma)^{2m+1} \in \left\langle [(k+\gamma)^{2i+1}] \colon 0 \le 2i+1 < \deg a(k) - 1 \right\rangle,$$

where

$$\gamma = -\beta + \frac{\alpha-1}{2}.$$

Proof. The proof is parallel to the proof of Theorem 3.2. Instead of (3.2), we take

$$x(k) = x_s(k) = \left(k + \gamma - \frac{1}{2}\right)^s$$

in Lemma 2.2. By Lemma 4.1, $(a(k), b(k))$ is not degenerated and

$$\deg(a(k) - b(k-1)) = \deg a(k) - 1.$$

Hence the polynomial

$$p_s(k) = a(k)x_s(k+1) - b(k-1)x_s(k)$$

satisfies

$$\deg p_s(k) = s + \deg a(k) - 1.$$

Moreover, we have

$$p_s(-\gamma - k) = (-1)^{s+1} p_s(-\gamma + k),$$

so that the reduction process maintains the symmetric property. Therefore, the reduction process continues until the degree is less than $\deg a(k) - 1$. ∎

Similar to Theorem 3.3, we have the following result.

Theorem 4.3 *Let*

$$t_k = \left(\frac{(\alpha)_k}{k!} \right)^r,$$

where r is a positive integer and α is a rational number with denominator D. Suppose that $-\alpha r \notin \mathbb{N}$. Then for any positive integer m, there exist integers $a_0, \ldots, a_{r-2}$ and a polynomial $x(k) \in \mathbb{Z}[k]$ such that

$$(2Dk+D\alpha)^m t_k = \frac{1}{C_m} \sum_{i=0}^{r-2} a_i (2Dk+D\alpha)^i t_k + \frac{1}{C_m} \Delta_k \left(2^{r-1} (Dk)^r x(2Dk) t_k \right),$$

where

$$C_m = \prod_{0 \le 2i \le m-r+1} ((\alpha r + m - r + 1 - 2i) \cdot D).$$

Moreover, $a_i = 0$ if $i \not\equiv m \pmod 2$.

Proof. The proof is parallel to the proof of Theorem 3.3. Instead of (3.6) and (3.7), we take

$$\tilde{x}_s(k) = (k + D\alpha - D)^s \tag{4.1}$$

and

$$\tilde{p}_s(k) = \frac{1}{2}((k + D\alpha)^r (k + D)^s - (k - D\alpha)^r (k - D)^s), \tag{4.2}$$

so that (3.5) still holds. It is clear that $\tilde{x}_s(k), \tilde{p}_s(k) \in \mathbb{Z}[k]$. But in this case, $\tilde{p}_s(k)$ is not monic. The leading term of $\tilde{p}_s(k)$ is

$$(\alpha r + s)D \cdot k^{s+r-1}.$$

Now let us consider the reduction process. Let $p(k) \in \mathbb{Z}[k]$ be a polynomial of degree $\ell \geq r-1$. Assume further that $p(k)$ contains only even powers of k or only odd powers of k. Setting

$$\begin{aligned} p'(k) &= \operatorname{lc}\tilde{p}_{\ell-r+1}(k) \cdot p(k) - \operatorname{lc} p(k) \cdot \tilde{p}_{\ell-r+1}(k) \\ &= (\alpha r + \ell - r + 1)D \cdot p(k) - \operatorname{lc} p(k) \cdot \tilde{p}_{\ell-r+1}(k), \end{aligned}$$

we see that $p'(k) \in \mathbb{Z}[k]$ and $\deg p'(k) < \ell$. Since $\tilde{p}_{\ell-r+1}(k)$ contains only even powers of k or only odd powers of k, so does $p'(k)$. Therefore, $\deg p'(k) \leq \ell - 2$.

Continuing this reduction process until $\ell < r-1$, we finally obtain that there exist integers $c_m, c_{m-2}, \ldots$ such that

$$C_m k^m - c_m \tilde{p}_{m-r+1}(k) - c_{m-2}\tilde{p}_{m-r-1}(k) - \cdots,$$

is a polynomial of degree less than $r-1$ and with integral coefficients, where C_m is the product of the leading coefficient of $\tilde{p}_{m-r+1}(k), \tilde{p}_{m-r-1}(k), \ldots$

$$C_m = \prod_{0 \leq 2i \leq m-r+1} \left((\alpha r + m - r + 1 - 2i)D\right),$$

as desired. ∎

For the special case of $t_k = (1/2)_k^4/(1)_k^4$, we may further reduce the factor C_m.

Lemma 4.4 *Let m be a positive integer and*

$$t_k = \frac{(1/2)_k^4}{(1)_k^4}.$$

- *If m is odd, then there exist an integer c and a polynomial $x(k) \in \mathbb{Z}[k]$ such that*

$$(4k+1)^m t_k = \frac{c}{C'_m}(4k+1)t_k + \frac{1}{C'_m}\Delta_k\left(32k^4 x(4k) t_k\right),$$

  where $C'_m = (\frac{m-1}{2})!$.

- *If m is even, then there exist integers c, c' and a polynomial $x(k) \in \mathbb{Z}[k]$ such that*

$$(4k+1)^m t_k = \frac{1}{C'_m}(c + (4k+1)^2 c')t_k + \frac{1}{C'_m}\Delta_k\left(64k^4 x(4k) t_k\right),$$

  where $C'_m = (m-1)!!$.

Proof. This is the special case of Theorem 4.3 for $\alpha = 1/2$ and $r = 4$. Therefore, $D = 2$ and $\alpha r - r + 1 = -1$.

We need only to show that the coefficients of $\tilde{p}_s(k)$ given by (4.2) is divisible by 2 when s is odd and is divisible by 4 when s is even. Then we may replace $\tilde{x}_s(k)$ given by (4.1) by $\tilde{x}_s(k)/4$ and $\tilde{x}_s(k)/2$ so that the leading coefficient of $\tilde{p}_s(k)$ is reduced. Correspondingly, the product C_m of the leading coefficients becomes

$$\prod_{0\le 2i\le m-3} \frac{1}{2}\,\mathrm{lc}\,\tilde{p}_{m-3-2i}(k) = \prod_{0\le 2i\le m-3} (m-1-2i) = (m-1)!!, \quad m \text{ even},$$

and

$$\prod_{0\le 2i\le m-3} \frac{1}{4}\,\mathrm{lc}\,\tilde{p}_{m-3-2i}(k) = \prod_{0\le 2i\le m-3} \frac{m-1-2i}{2} = \left(\frac{m-1}{2}\right)!, \quad m \text{ odd}.$$

Notice that

$$\tilde{p}_s(k) = \frac{1}{2}((k+1)^4\,(k+2)^s - (k-1)^4\,(k-2)^s).$$

The coefficient of k^j is

$$\frac{1-(-1)^{s-j}}{2} \sum_{0\le \ell\le 4,\ 0\le j-\ell\le s} \binom{4}{\ell}\binom{s}{j-\ell} 2^{s-j+\ell}.$$

If $j-\ell < s$, the corresponding summand is divisible by 2. If $j-\ell = s$ and ℓ is even, then $(-1)^{s-j} = 1$ and the coefficient is 0. Otherwise, $\ell = 1$ or $\ell = 3$, and thus $4 \mid \binom{4}{\ell}$. Therefore, the coefficient must be divisible by 2.

Now consider the case of s being even. If $j-\ell < s-1$, the corresponding summand is divisible by 4. Otherwise $j-\ell = s$ or $j-\ell = s-1$. We have seen that if $j-\ell = s$, then the coefficient is divisible by 4. If $j-\ell = s-1$. Then

$$\binom{s}{j-\ell} = s \quad \text{and} \quad 2^{s-j+\ell} = 2.$$

Thus the summand is also divisible by 4. ∎

Example 4.2 *Consider the case of $m = 11$. We have*

$$(4k+1)^{11}t_k + 10515(4k+1)t_k = \Delta_k(32k^4 p(k)t_k)$$

where

$$p(k) = \frac{1}{5}(4k-1)^8 - \frac{249}{20}(4k-1)^6 + \frac{20207}{60}(4k-1)^4 - \frac{89909}{20}(4k-1)^2 + \frac{524029}{20}.$$

As an application, we obtain the following congruences.

Theorem 4.5 *Let m be a positive odd integer and $\mu = (m-1)/2$. Denote*

$$S_m = \sum_{k=0}^{\frac{p-1}{2}} (4k+1)^m \left(\frac{(1/2)_k}{(1)_k} \right)^4.$$

Then there exists an integer a_m such that for each prime $p > \mu$,

$$S_m \equiv \frac{a_m}{\mu!} p \pmod{p^4}.$$

Proof. By Lemma 4.4, there exist an integer a_m and a polynomial $q_m(k) \in \mathbb{Z}[k]$ such that

$$(4k+1)^m t_k - \frac{a_m}{\mu!}(4k+1)t_k = \frac{1}{\mu!}\Delta_k \left(32k^4 q_m(k) t_k\right),$$

where $t_k = \left(\frac{(1/2)_k}{(1)_k} \right)^4$. Summing over k from 0 to $(p-1)/2$, we obtain

$$S_m - \frac{a_m}{\mu!} S_1 = 32\omega^4 \frac{q_m(4\omega)}{\mu!} \left(\frac{(1/2)_\omega}{(1)_\omega} \right)^4,$$

where $\omega = (p+1)/2$. When $p > \mu$, $1/\mu!$ is a p-adic integer and

$$\left(\frac{(1/2)_\omega}{(1)_\omega} \right)^4 \equiv 0 \pmod{p^4}.$$

Therefore,

$$S_m \equiv \frac{a_m}{\mu!} S_1 \pmod{p^4}.$$

It is shown by Long [12] that

$$S_1 \equiv p \pmod{p^4},$$

completing the proof. ∎

The integer a_m and the polynomial $q_m(k)$ can be computed by the extended Zeilberger's algorithm.

By checking the initial values, we propose the following conjecture.

Conjecture 4.6 *For any positive odd integer m, the coefficient $a_m/(\frac{m-1}{2})!$ is an integer.*

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