Self Avoiding Walks, The Language[1] of Science, and Fibonacci Numbers

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Science is a language. In fact, science *is* language. This was shown brilliantly by Xavier Viennot and his *Ecole Bordelaise* (e.g. [V][DV][B]). Viennot, Maylis Delest, and their disciples code *animals* and other *physical* creatures in terms of *algebraic* (context-free) *languages*, by using the so-called *Schutzenberger methodology* (which Marco Schutzenberger prefers to call the *DSV* (Dyck-Schutzenberger-Viennot) methodology.)

In this note, I use this philosophy, or rather a juvenile version of it, to enumerate *self avoiding walks* in the (discrete) region $\{0,1\} \times [-\infty, \infty]$, by encoding these walks in terms of *words* in a certain *rational* ("finite- automata") language, that I call the "UL^*IU' language", and by describing its syntax.

A *self avoiding walk* (saw) in the two-dimensional (square) lattice is a finite sequence of *distinct* lattice points $[(x_0, y_0) = (0,0), (x_1, y_1), \dots, (x_n, y_n)]$, such that for all i, (x_i, y_i) and (x_{i+1}, y_{i+1}) are *nearest neighbors*. The nearest neighbors of a point (a,b) are the four points $(a+1,b)$,$(a-1,b)$,$(a,b+1)$,$(a,b-1)$. The problem of finding the exact, and even asymptotic, value of a_n, the number of saws with n steps, is wide open, and presumably very difficult. The analogous problem in dimensions > 4, for the asymptotics, was recently solved brilliantly by Hara and Slade[HS], and beautifully exposited in the masterpiece by Madras and Slade[MS].

When a problem seems intractable, it is often a good idea to try to study "toy" versions of it in the hope that as the toys become increasingly larger and more sophisticated, they would metamorphose, in the limit, to the *real thing*. That was essentially Lars Onsager's[O](see[T]) way of solving the Ising model. Onsager first solved the "finitary" Ising problem in a strip of finite-width, that turned out to be a finite (definite) sum, that miraculously converged, a la Riemann-Integral, to a certain definite integral.

Alm and Janson[AJ] had a similar idea of approaching general saws by studying saws on strips $[-L, M] \times [-\infty, \infty]$, with L and M finite. Saws, when viewed "dynamically", are the epitome of non-Markovianess. In [AJ] it was observed that when saws are viewed "statically", and restricted to a strip, they can be described as Markov Chains. A saw can be viewed statically, since the path a self-avoiding drunkard makes uniquely determines her (or his) history. The general saw can be similarly viewed as a "Markov chain", but this time the number of states is infinite. Since it is much easier to describe the states then the saws themselves, there is some hope that, by replacing the transition matrices by suitable operators on some Hilbert (or whatever) space, this approach will conquer the general problem. Only now we transcend the rational, and even algebraic paradigms,

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[1] This paper is dedicated to the memory of my father Yehudah Heinz Zeilberger(1915-1994), who spoke, read, and wrote fluently in seven (natural) languages.

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into the holonomic paradigm and beyond.

Alm and Jansen's[AJ] motivation was numerical. They wanted to find lower bounds for the *connective constant*, μ ($:= \lim_{n\to\infty} a_n^{1/n}$), by computing the corresponding connective constants for saws in increasingly wider strips. These turn out to be eigenvalues of matrices with integer entries, and hence algebraic numbers.

Myself, I care little for *real*, floating-point numbers. Being a discretian, I strive to get the *exact* answer. The theorem below gives an exact enumeration of n-step saws in the strip $\{0,1\}\times[-\infty,\infty]$. More interesting than the result is the *linguistic method of proof*, that would hopefully generalize.

Theorem: The number, $a_n^{(2)}$, of $n-$step saws in the strip $\{0,1\}\times[-\infty,\infty]$ is given by $a_0^{(2)}=1$, $a_1^{(2)}=3$, and for $n>1$, by

$$a_n^{(2)} = 8F_n - \frac{n}{2}(1+(-1)^n) - 2(1-(-1)^n) \quad .$$

Proof: We assume that readers are familiar with the language of *generatingfunctionlogy*[W].[3] From now on, let gf stand for "(ordinary) generating function".

Any saw in $\{0,1\}\times[-\infty,\infty]$ has the form UL^*IU', where the meanings of U, L, I, U' are as follows. (Steps in the right, left, up, and down direction will be denoted by r, l, u, and d respectively. For example the walk $(0,0),(0,1),(0,2),(1,2),(1,1)$ will be coded as $uurd$. Also d^i means $dd\ldots d$, where d is repeated i times.[4])

(i) U is a U-turn: d^iru^i, with $i\geq 0$ ($i=0$ corresponds to a degenerate U-turn) (gf$=t/(1-t^2)$), or nothing (gf=1). Total gf for this part is $1+t/(1-t^2)$.

(ii) L^*: Any number of (upside-down)Ls (or Γs), interlaced with upside-down-dyslectic Ls. A single L is either u^il or u^ir ($i\geq 1$). The gf of a single L is $t^2/(1-t)$, and hence that of L^* is $1/(1-[t^2/(1-t)])=(1-t)/(1-t-t^2)$. (Philofibonaccist rejoice!)

(iii) an I, or nothing: $u^i, i\geq 0$. Its gf is $1/(1-t)$.

(iv) A final, modified U-turn, that I call U', which is $u^{i+1}ld^i$, or $u^{i+1}rd^i$, $i\geq 1$ (gf$=t^4/(1-t^2)$), or nothing (gf=1). The total gf is $1+t^4/(1-t^2)$.

The gf for the combined words UL^*IU' is thus:

$$[1+\frac{t}{(1-t^2)}]\cdot[\frac{(1-t)}{(1-t-t^2)}]\cdot[\frac{1}{(1-t)}]\cdot[1+\frac{t^4}{(1-t^2)}] = \frac{(1+t-t^2)(1-t^2+t^4)}{(1-t^2)^2(1-t-t^2)} \quad .$$

[3] Buy your own copy today! It would cost you less than 1/4 cent per day (Gian-Carlo Rota, in the "Bulletin for Mathematics books and software", states that "this book is good for the next fifty years".)

[4] Puzzle: What word in the English language has the largest number of double letters? Ans: $sub^2o^2k^2e^2per$

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But this is only half of the story: the northbound walks. By symmetry, the gf of the other half, the southbound walks, which are the x-axis mirror-reflection of the first half, is the same. But two walks have been counted twice: the 0-step empty walk (gf=1), and the 1-step walk $[(0,0),(1,0)] = r$ (gf=t). So the final gf is twice the gf above, take away $1+t$, namely

$$2\frac{(1+t-t^2)(1-t^2+t^4)}{(1-t^2)^2(1-t-t^2)} - (1+t) = \frac{1+2t-t^3-t^4+t^7}{(1-t)^2(1+t)^2(1-t-t^2)} \quad .$$

A partial-fraction decomposition (that Maple^{TM} kindly performed for me), followed by a Maclaurin expansion, yields the formula for $a_n^{(2)}$. $\square$

A Shorter, more elegant, Semi-Rigorous, late-21st Century-Style Proof: Compute $a_n^{(2)}$ by direct enumeration for $0 \leq n \leq 15$, and then use Salvy and Zimmerman's[SZ] Maple package *gfun* to conjecture the gf. Since we know a priori that this is a rational function, that must be it. $\square$

To make this argument completely rigorous, you would have to derive a priori bounds for the degrees of the numerator and denominator of the gf, but *who cares ?*

The only possible advantage of the first proof is that it might generalize to obtain the gfs, $\phi_r(t)$ for the number of saws in the strip $[-r,r] \times [-\infty,\infty]$, for $r = 1, 2, \ldots$. Of course, the expressions themselves will very soon become unwieldy. More exciting is the prospect that one might be able to find some kind of functional equation that expresses $\phi_r(t)$ in terms of $\phi_{r-1}(t)$, or more refined quantities, from which the divine quantity $\phi(t) := \lim_{r\to\infty} \phi_r(t)$ could be looked at in the eyes, without being blinded. *Amen ken yehi ratson.*

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