A WZ PROOF OF RAMANUJAN'S FORMULA FOR π

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Dedicated to Archimedes on his 2300^{th} *birthday*

Archimedes computed π very accurately. Much later, Ramanujan discovered several infinite series for $1/\pi$ that enables one to compute π even more accurately. The most impressive one is([Ra]): ($(a)_k$ denotes, as usual, $a(a+1)...(a+k-1)$.)

$$\frac{1}{\pi} = 2\sqrt{2}\sum_{k=0}^{\infty} \frac{(1/4)_k(1/2)_k(3/4)_k}{k!^3}(1103+26390k)(1/99)^{4k+2}. \tag{1}$$

This formula is an example of a *non-terminating* hypergeometric series identity. Many times, non-terminating series are either limiting cases or "analytic continuations" of *terminating identities*, which are now known to be routinely provable by computer. [WZ].

While we do not know of a terminating generalization of (1), we do know how to give a WZ proof of another formula for π, also given by Ramanujan[Ra], and included in his famous letter to Hardy. This formula is:

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{k!^3} \quad . \tag{2}$$

The terminating version, that we will prove is

$$\frac{\Gamma(3/2+n)}{\Gamma(3/2)\Gamma(n+1)} = \sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^2(-n)_k}{k!^2(3/2+n)_k}. \tag{3}$$

To prove it for all *positive* integers n, we call the summand divided by the left side $F(n,k)$, and cleverly construct

$$G(n,k) := -\frac{(2k+1)^2}{(2n+2k+3)(4k+1)}F(n,k)\,,$$

with the motive that $F(n+1,k) - F(n,k) = G(n,k) - G(n,k-1)$ (check!), and summing this last identity w.r.t k shows that $\sum_k F(n,k) \equiv Constant$, which is seen to be 1, by plugging in $n=0$. This proves (3). To deduce (2), we "plug" in $n = -1/2$, which is legitimate in view of Carlson's theorem [Ba].

REFERENCES

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