THE QUANTUM MACMAHON MASTER THEOREM

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ABSTRACT. We state and prove a quantum-generalization of MacMahon's celebrated Master Theorem, and relate it to a quantum-generalization of the boson-fermion correspondence of Physics.

1. INTRODUCTION

1.1. **MacMahon's Master Theorem.** In this paper we state and prove a quantum-generalization of MacMahon's celebrated Master Theorem, conjectured by the first two authors. Our result was motivated by quantum topology. In addition to its potential importance in knot theory and quantum topology (explained in brief in the last section), this paper answers George Andrews's long-standing open problem [A] of finding a *natural* q-analog of MacMahon's Master Theorem.

Let us recall the original form of MacMahon's Master Theorem and some of its modern interpretations.

Consider a square matrix $A = (a_{ij})$ of size r with entries in some commutative ring. For $1 \leq i \leq r$, let $X_i := \sum_{j=1}^r a_{ij} x_j$, (where x_i's are commuting variables) and for any vector $(m_1, \dots, m_r)$ of non-negative integers let $G(m_1, \dots, m_r)$ be the coefficient of $x_1^{m_1} x_2^{m_r} \dots x_r^{m_r}$ in $\prod_{i=1}^r X_i^{m_i}$. *MacMahon's Master Theorem* is the following identity (see [MM]):

$$\sum_{m_1, m_2, \dots, m_r = 0}^{\infty} G(m_1, \dots, m_r) = 1/\det(I - A). \tag{1}$$

There are several equivalent reformulations of MacMahon's Master Theorem; see for example [FZ] and references therein. Let us mention one, of importance to physics.

Given a matrix $A = (a_{ij})$ of size r with commuting indeterminate entries which lie in a ring $\mathcal{R}$, and a nonnegative integer n, we can consider its *symmetric* and *exterior* powers $S^n(A)$ and $\Lambda^n(A)$, and their traces $\operatorname{tr} S^n(A)$ and $\operatorname{tr} \Lambda^n(A)$ respectively. Since

$$\begin{aligned} \operatorname{tr} S^n(A) &= \sum_{m_1 + \dots m_r = n} G(m_1, \dots, m_r) \\ \det(I - A) &= \sum_{n=0}^{\infty} (-1)^n \operatorname{tr} \Lambda^n(A) t^n, \end{aligned}$$

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the following identity

$$\frac{1}{\sum_{n=0}^{\infty}(-1)^n \operatorname{tr} \Lambda^n(A)t^n} = \sum_{n=0}^{\infty} \operatorname{tr} S^n(A)t^n \tag{2}$$

in $\mathcal{R}[[t]]$ is equivalent to (1). In Physics (2) is called the *boson-fermion* correspondence, where bosons (resp. fermions) are commuting (resp. skew-commuting) particles corresponding to symmetric (resp. exterior) powers.

1.2. **Quantum algebra, right-quantum matrices and quantum determinants.** In r-dimensional *quantum algebra* we have r indeterminates x_i ($1 \leq i \leq r$), satisfying the commutation relations $x_j x_i = q x_i x_j$ for all $1 \leq i < j \leq r$. We also consider matrices $A = (a_{ij})$ of r^2 indeterminates a_{ij}, $1 \leq i, j \leq r$, that commute with the x_i's and such that for any 2 by 2 minor of (a_{ij}), consisting of rows i and i', and columns j and j' (where $1 \leq i < i' \leq r$, and $1 \leq j < j' \leq r$), writing $a := a_{ij}, b := a_{ij'}, c := a_{i'j}, d := a_{i'j'}$, we have the *commutation relations*:

$$ca = qac, \quad (q\text{-commutation of the entries in a column}) \tag{3}$$

$$db = qbd, \quad (q\text{-commutation of the entries in a column}) \tag{4}$$

$$ad = da + q^{-1}cb - qbc \qquad (\text{cross commutation relation}). \tag{5}$$

We will call such matrices A *right-quantum matrices.*

The *quantum determinant*, (first introduced in [FRT]) of any (not-necessarily right-quantum) r by r matrix $B = (b_{ij})$ may be defined by

$$\det_q(B) := \sum_{\pi \in S_r} (-q)^{-\mathrm{inv}(\pi)} b_{\pi_1 1} b_{\pi_2 2} \cdots b_{\pi_r r},$$

where the sum ranges over the set of permutations, S_r, of $\{1, \ldots, r\}$, and for any of its members, π, $\mathrm{inv}(\pi)$ denotes the number of pairs $1 \leq i < j \leq r$ for which $\pi_i > \pi_j$.

1.3. **A q-version of MacMahon's Master Theorem.** We are now ready to state our quantum version of Macmahon's Master Theorem.

Theorem 1. (Quantum MacMahon Master Theorem) *Fix a right-quantum matrix A of size r. For $1 \leq i \leq r$, let $X_i := \sum_{j=1}^r a_{ij} x_j$, and for any vector $(m_1, \ldots, m_r)$ of non-negative integers let $G(m_1, \ldots, m_r)$ be the coefficient of $x_1^{m_1} x_2^{m_r} \ldots x_r^{m_r}$ in $\prod_{i=1}^r X_i^{m_i}$. Let*

$$\mathrm{Ferm}(A) = \sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \det_q(A_J)$$

where A_J is the J by J submatrix of A, and

$$\mathrm{Bos}(A) = \sum_{m_1, \ldots, m_r = 0}^{\infty} G(m_1, \ldots, m_r).$$

Then

$$\mathrm{Bos}(A) = 1/\mathrm{Ferm}(A).$$

When we specialize to $q = 1$, then Theorem 1 becomes the (2), which explains why our result is a q-version of the MacMahon Master Theorem. For a motivation of Theorem 1, see Section 3.

The above result is not only interesting from the combinatorial point of view, but it is also a key ingredient in a *finite noncommutative formula* for the colored Jones function of a knot. This will be explained in a subsequent publication, [GL].

1.4. **Computer code.** The results of the paper have been verified by computer code, written by the third author. `Maple` programs `QuantumMACMAHON` and `qMM` are available at: `http://www.math.rutgers.edu/`
`~zeilberg/`. The former proves rigorously Theorem 1 for any fixed r.

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2. Proof

2.1. **Some lemmas on operators.** The proof will make crucial use of a *calculus of difference operators*, developed by the third author in [Z1]. This calculus of difference operators predates the more advanced calculus of holonomic functions, developed by the third author in [Z2].

Difference operators act on *discrete functions* F, that is functions whose domain is $\mathbb{N}^r$. For example, consider the *shift-operators* M_i and the *multiplication* operator Q_i which act on a discrete function $F(m_1, \dots, m_r)$ by

$$\begin{aligned} (M_iF)(m_1, \dots, m_r) &:= F(m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_r) \\ (Q_iF)(m_1, \dots, m_r) &:= q^{m_i} F(m_1, \dots, m_r). \end{aligned}$$

It is easily seen that

$$M_iQ_i = qQ_iM_i.$$

Abbreviating Q_i by q^{m_i}, we obtain that:

$$M_i q^{m_i} = q^{m_i+1} M_i \qquad M_i q^{m_j} = q^{m_j} M_i \quad \text{for} \quad i \neq j. \tag{6}$$

Another example is the operator $\hat{x}_i$ which left multiplies F by x_i. Notice that $\hat{x}_j\hat{x}_i = q\hat{x}_i\hat{x}_j$ for $j > i$. In the proof below, we will denote $\hat{x}_i$ by x_i. In that case, the identity $x_jx_i = qx_ix_j$ for $j > i$ holds in the quantum algebra, as well as in the algebra of operators.

Before embarking on the proof, we need the following readily-verified lemmas.

Lemma 2.1. *(commuting X_i with X_j) For $1 \leq i < j \leq r$, $X_jX_i = qX_iX_j$.*

Lemma 2.2. *(commuting x_i with X_j) For each of the a_{ij}, define the operator Q_{ij} acting on expressions P involving a_{ij} by $Q_{ij}P(a_{ij}) := P(qa_{ij})$. Then, for any $1 \leq i, j \leq r$, and integer m_i and any expression F*

$$x_i^{-m_i} X_j F = [(Q_{j1}^{-1} Q_{j2}^{-1} \cdots Q_{j,i-1}^{-1} Q_{j,i+1} \cdots Q_{jr})^{m_i} X_j] x_i^{-m_i} F.$$

Lemma 2.3. *(Column expansion with respect to the last column): Given an r by r matrix (a_{ij}) (not necessarily quantum) let A_i be the minor of the entry a_{ir}, i.e. the $r-1$ by $r-1$ matrix obtained by deleting the i^{th} row and r^{th} column. Then*

$$\det_q(A) = \sum_{i=1}^{r} (-q)^{i-r} (\det_q A_i)\, a_{ir}.$$

Lemma 2.4. *If A is a matrix that satisfies Equation (5) and A' denotes a matrix obtained by interchanging the i and j columns columns of A, then $\det_q(A') = (-q)^{-\mathrm{inv}(ij)} \det_q(A)$.*

Proof. Suppose first that we interchange two adjacent colums i and $j := i+1$. Consider the involution of S_r that sends a permutation π to $\pi' = \pi(ij)$. Given $\pi \in S_r$, let $(A;\pi) = (-1)^{\mathrm{inv}(\pi)} a_{\pi_1 1} \dots a_{\pi_r r}$ denote the contribution of π in $\det_q(A)$. Then, $\det_q(A) = \sum_\pi (A;\pi)$. Equation (5) implies that

$$(A;\pi) + (A;\pi') = (-q)((A';\pi) + (A';\pi')).$$

Summing over all permutations proves the result when $j = i+1$.

Observe that when $j = i+1$, the matrix A' is no longer right-quantum since it does not satisfy (5). However, the proof used only the fact that (5) holds for the i and $i+1$ columns of A.

Thus, the proof can be iterated $\mathrm{inv}(ij)$ times to commute the i and $j > i$ columns of A. The result follows. ☐

Lemma 2.5. *(Equal columns imply that* $\det_q$ *vanishes): Let A be a right-quantum matrix. In the notation of Lemma 2.3, for all $j \neq r$,*

$$\sum_{i=1}^{r} (-q)^{i-r} \, (\det_q A_i) \, a_{ij} = 0.$$

Proof. If $j = r-1$, it is easy to see that q-commutation along the entries in every column of A imply that the sum vanishes.

If $j < r-1$, use Lemma 2.4 to reduce it to the case of $j = r-1$. ☐

Remark 2.6. One can give an alternative proof of Lemmas 2.4 and 2.5 from the trivial 2 by 2 case and, by induction using the q-Laplace expansion of a q-determinant that is completely analogous to the classical case.

2.2. **Proof of Theorem 1.** The proof is a quantum-adaptation of the "operator-elimination" proof of MacMahon's Master Theorem given in [Z1]. Fix a right-quantum matrix A.

Observe that $G(m_1, \dots, m_r)$ is the coefficient of $x_1^0 \dots x_r^0$ in

$$H(m_1, \dots, m_r; x_1, \dots, x_r) := x_r^{-m_r} \cdots x_2^{-m_2} x_1^{-m_1} \prod_{i=1}^{r} X_i^{m_i}.$$

We will think of H as a *discrete function*, that is as a function of $(m_1, \dots, m_r) \in \mathbb{N}^r$. H takes values in the ring of noncommutative Laurrent polynomials in the x_is, with coefficients in the ring generated by the entries of A, modulo the ideal given by (3)-(5).

Let's see how the shift operators M_i acts on H. By definition,

$$M_i H(m_1, \dots, m_r; x_1, \dots, x_r) =$$
$$x_r^{-m_r} \cdots x_{i+1}^{-m_{i+1}} x_i^{-m_i - 1} x_{i-1}^{-m_{i-1}} \cdots x_1^{-m_1} X_1^{m_1} \cdots X_{i-1}^{m_{i-1}} X_i^{m_i+1} X_{i+1}^{m_{i+1}} \cdots X_r^{m_r}.$$

By moving ${x_i}^{-1}$ to the front and X_i in front of ${X_1}^{m_1}$, and using Lemma 2.1 and $x_j x_i = q x_i x_j$, we have

$$M_i H(m_1, \dots, m_r) = q^{m_r + m_{r-1} + \cdots + m_{i+1} - m_1 - m_2 - \cdots - m_{i-1}} x_i^{-1} [x_r^{-m_r} \cdots x_1^{-m_1} X_i] X_1^{m_1} \cdots X_r^{m_r}.$$

By moving X_i next to x_i^{-1} and using Lemma 2.2 this equals to:

$$q^{m_r + m_{r-1} + \cdots + m_{i+1} - m_1 - m_2 - \cdots - m_{i-1}} x_i^{-1} \cdot$$
$$[(Q_{i2} \cdots Q_{ir})^{m_1} (Q_{i1}^{-1} Q_{i3} \cdots Q_{ir})^{m_2} (Q_{i1}^{-1} Q_{i2}^{-1} Q_{i4} \cdots Q_{ir})^{m_3} \cdots (Q_{i1}^{-1} Q_{i2}^{-1} \cdots Q_{i,r-1}^{-1})^{m_r} X_i] \cdot$$
$$x_r^{-m_r} \cdots x_1^{-m_1} {X_1}^{m_1} \dots {X_r}^{m_r},$$

which is equal to

$$q^{m_r+m_{r-1}+\cdots+m_{i+1}-m_1-m_2-\cdots-m_{i-1}}x_i^{-1}\cdot$$

$$\left(q^{-m_2-m_3-\cdots-m_r}a_{i1}x_1+q^{m_1-m_3-\cdots-m_r}a_{i2}x_2+\cdots+q^{m_1+m_2+\cdots+m_{r-1}}a_{ir}x_r\right)H(m_1,\ldots,m_r;x_1,\ldots,x_r)$$

Multiplying out and rearranging, we get that the discrete function $H(m_1,\ldots,m_r;x_1,\ldots,x_r)$ is annihilated by the r operators $(i=1,2,\ldots,r)$

$$\mathcal{P}_i:=\sum_{j=1}^{i-1}-q^{-m_j-2m_{j+1}-\cdots-2m_{i-1}-m_i}a_{ij}x_j+(M_i-a_{ii})x_i+\sum_{j=i+1}^{r}-q^{m_i+2m_{i+1}+\cdots+2m_{j-1}+m_j}a_{ij}x_j.$$

Now comes a nice surprise. Let us define b_{ij} to be the coefficient of x_j in $\mathcal{P}_i$. For example, for $r=3$ we have:

$$B=\begin{pmatrix} M_1-a_{11} & -q^{m_1+m_2}a_{12} & -q^{m_1+2m_2+m_3}a_{13} \\ -q^{-m_1-m_2}a_{21} & M_2-a_{22} & -q^{m_2+m_3}a_{23} \\ -q^{-m_1-2m_2-m_3}a_{13} & -q^{-m_2-m_3}a_{23} & M_3-a_{33}\end{pmatrix}.$$

Lemma 2.7. *B is a right-quantum matrix.*

Proof. It is easy to see that the entries in each column of B q-commute. To prove Equation (5), consider the following cases for a 2 by 2 submatrix C of B: C contains two, (resp. one, resp. no) diagonal entries of B, and prove it case by case, using the fact that the operators M_i and q^{m_j} commute with the a_{ij}, and satisfy the commutation relations (6). $\square$

Now we eliminate $x_1,x_2,\ldots,x_{r-1}$ by left-multiplying $\mathcal{P}_i$ by the minor of b_{ir} in $B=(b_{ij})$ times $(-q)^{i-r}$, for each $i=1,2,\ldots,r$, and adding them all up. Since B is right-quantum (by Lemma 2.7), Lemma 2.5 implies that the coefficients of $x_1,\ldots,x_{r-1}$ all vanish, and $\det_q(B)x_rH=0$. After left multiplying by x_r^{-1} which commutes with the entries in B, we obtain that

$$\det_q(B)H(m_1,\ldots,m_r;x_1,\ldots,x_r)=0.$$

Since the entries of B do not contain x_i's, it follows that $\det_q(B)$ annihilates each and every single coefficient of H, in particular its constant term. Taking the constant term yields

$$\det_q(B)G(m_1,\ldots,m_r)=0.$$

Here comes the next surprise.

Lemma 2.8. (a) We have:

$$\det_q(B)=\sum_{J\subset\{1,\ldots,r\}}(-1)^{|J|}\det_q(A_J)M_{\bar{J}}$$

where $\bar{J}=\{1,\ldots,r\}-J$ and $M_J=\prod_{j\in J}M_j$.
(b) In particular,

$$\det_q(B)|_{M_1=\cdots=M_r=1}=\mathrm{Ferm}(A).$$

Proof. Let us expand $\det_q(B)$ as a sum over permutations $\pi\in S_r$. We have:

$$\begin{aligned}\det_q(B) &= \sum_{\pi\in S_r}(-q)^{-\mathrm{inv}(\pi)}b_{\pi_1 1}b_{\pi_2 2}\cdots b_{\pi_r r} \\ &= \sum_{\pi\in S_r}\prod_{i=1}^{r}(-q)^{-\mathrm{inv}(\pi,i)}b_{\pi_i i}\end{aligned}$$

where $\mathrm{inv}(\pi, i)$ is the number of $j > i$ such that $\pi_i > \pi_j$. Now, $b_{ij} = q_{ij}a_{ij}$ for $i \neq j$, where q_{ij} is a monomial in the variables q^{m_i}, and $\prod_j q_{ij} = 1$. Moreover, the exponent of q^{m_i} in q_{ij} is 2 if $j > i$ and -2 if $j < i$.

Since $\prod_j q_{ij} = 1$, we can move the monomials q_{ij} in the left of $\prod_i (-q)^{\mathrm{inv}(\pi,i)} b_{\pi_1 1} \dots b_{\pi_r r}$, and then cancel them. The monomials commute with all entries of the matrix b_{ij}, *except* with the diagonal ones. Commuting q^{2m_i} with $b_{ii} = \delta_{\pi_i i} M_{ii} - a_{ii}$ gives: $b_{ii} q^{2m_i} = q^{2m_i}(\delta_{\pi_i i} q^2 M_{ii} - a_{ii})$. In other words, it replaces M_i by $q^2 M_i$. Thus, we have:

$$\begin{aligned}
\det_q(B) &= \sum_{\pi \in S_r} \prod_{i=1}^{r} (-q)^{-\mathrm{inv}(\pi,i)} (\delta_{\pi_i i} q^{2\mathrm{inv}(\pi,i)} M_i - a_{\pi_i i}) \\
&= \sum_{\pi \in S_r} \sum_{J \subset \{1,\dots,r\}} \prod_{i \in J} (-q)^{-\mathrm{inv}(\pi,i)} \delta_{\pi_i i} q^{2\mathrm{inv}(\pi,i)} M_i \prod_{i \notin J} (-q)^{-\mathrm{inv}(\pi,i)} (-a_{\pi_i i})
\end{aligned}$$

Now, rearrange the summation. Observe that every permutation π of $\{1, \dots, r\}$ gives rise to a permutation π' on the set $\{1, \dots, r\} - \mathrm{Fix}(\pi)$, where $\mathrm{Fix}(\pi)$ is the fixed point set of π. Moreover, $\mathrm{inv}(\pi', i) = \mathrm{inv}(\pi, i) - |\{j \in J : j > i\}|$. Using this, part (a) follows. Part (b) follows from part (a) and the definition of $\mathrm{Ferm}(A)$. $\square$

Hence

$$\sum_{J \subset \{1,\dots,r\}} (-1)^{|J|} \det_q(A_J) M_{\bar{J}} G(m_1, \dots, m_r) = 0.$$

Summing over $\mathbb{N}^r$, we get:

$$\sum_{m_1,\dots,m_r=0}^{\infty} \sum_{J \subset \{1,\dots,r\}} (-1)^{|J|} \det_q(A_J) M_{\bar{J}} G(m_1, \dots, m_r) = 0.$$

For a subset $J = \{k_1, \dots, k_j\}$ of $\{1, \dots, r\}$, we denote by $G_J(m_{k_1}, \dots, m_{k_j})$ the evalutation $G(m_1, \dots, m_r)$ at $m_i = 0$ for all $i \notin J$, and we define

$$S_J = \sum_{m_{k_1},\dots,m_{k_j}=0}^{\infty} G(m_1, \dots, m_r).$$

Using telescoping cancellation, the inclusion-exclusion principle, and Lemma 2.8(b), the above equation becomes

$$\sum_{J \subseteq \{1,\dots,r\}} (-1)^{|J|} \mathrm{Ferm}(A_J) S_J = 0.$$

Using induction (with respect to r), together with $S_\emptyset = 1$, we obtain that $\mathrm{Ferm}(A) S_{\{1,\dots,r\}} = 1$. This concludes the proof of the theorem. $\square$

3. Some remarks on the boson-fermion correspondence

Let us give some motivation for Theorem 1 from the point of view of quantum topology.

For a reference on quantum space and quantum algebra, see [Ka, Chapter IV] and [M].

Recall that a vector (column or row) of r indeterminate entries $x_1, \dots, x_r$ lies in r-dimensional *quantum space* $A^{r|0}$ iff its entries satisfy

$$x_j x_i = q x_i x_j$$

for all $1 \leq i < j \leq r$.

Recall that a *right* (resp. *left*) *endomorphism* of $A^{r|0}$ is a matrix $A=(a_{ij})$ of size r whose entries commute with the coordinates x_i of a vector $x=(x_1,\dots,x_r)^T\in A^{r|0}$ and in addition, Ax (resp. x^TA) lie in $A^{r|0}$. Recall also that an endomorphism of $A^{r|0}$ is one that is right and left endomorphism.

It is easy to see (eg. in [Ka, Thm. IV.3.1]) that A is a right-quantum (i.e., a right-endomorphism) iff for every 2 by 2 submatrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ of A we have:

$$ca = qac, \qquad db = qbd, \qquad ad = da + q^{-1}cb - qbc.$$

Moreover, A is left-quantum iff for every 2 by 2 submatrix of A (as above) we have:

$$ba = qab, \qquad dc = qcd, \qquad ad = da + q^{-1}bc - qcb.$$

Finally, A is quantum iff for every 2 by 2 submatrix of A (as above) we have:

(7)

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \quad ad = da + q^{-1}cb - qbc.$$

The set of quantum matrices A are the points of the r-dimensional *quantum algebra* $M_q(r)$, which is defined to be the quotient of the free algebra in noncommuting variables x_{ij} for $1\le i,j,\le r$, modulo the left ideal generated by the commutation relations of Equation (7).

The algebra $M_q(r)$ has interesting and important structure. $M_q(r)$ is Noetherian, has no zero divisors, and in addition, a basis for the underlying vector space is given by the set of *sorted monomials* $\{\prod_{i,j} a_{ij}^{n_{ij}} | n_{ij}\ge 0\}$ where the product is taken lexicographically; see [Ka, Thm IV.4.1]. An important quotient of $M_q(r)$ is the *quantum group* $SL_q(r) := M_q(r)/(\det_q - 1)$, which is a Hopf algebra [Ka, Sec.IV.6] whose representation theory gives rise to the quantum group invariants of knots, such as the celebrated *Jones polynomial.*

Observing that

$$\begin{aligned}\operatorname{tr} S^n(A) &= \sum_{m_1+\dots m_r=n} G(m_1,\dots,m_r)\\ \operatorname{tr}\Lambda^n(A) = &= \sum_{J\subset\{1,\dots,r\},|J|=n} \det_q(A_J)\end{aligned}$$

Theorem 1 implies that:

Theorem 2. *If A is in $M_q(r)$, then*

$$\frac{1}{\mathrm{Ferm}(A)} = \sum_{n=0}^{\infty} \operatorname{tr} S^n(A)$$

Since the algebra $M_q(r)$ has a vector space basis given by sorted monomials, it should be possible to give an alternative proof of the quantum MacMahon Master Theorem using *combinatorics on words*, as was done in [FZ] for several proofs of the MacMahon Master theorem. We hope to return to this alternative point of view in the near future.

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