How to Extend Károlyi and Nagy's BRILLIANT Proof of the Zeilberger-Bressoud q-Dyson Theorem in order to Evaluate ANY Coefficient of the q-Dyson Product

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Dedicated to Freeman Dyson (b. Dec. 15, 1923) on his $89\frac{2}{3}$-th birthday

Very Important: As in all our joint papers, this document, the human-readable *article*, is not the main point, but its Maple implementation, `qDYSON`, written by DZ, available from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/qDYSON` .

The 'front' of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/qdyson.html` ,

contains sample input and output files, chuckfull of rigorously-derived deep identities, computed by SBE.

In fact, since we believe in *free open access*, we supply the *source code*, that, in principle, is also *humanly-readable*, but only if one knows Maple, and since most people do not know it fluently enough, DZ kindly prepared the present article.

Let's first do a redux of Gyula Károlyi and Zoltán Lóránt Nagy's *proof from the book*[KN], of the Zeilberger-Bressoud[ZB] theorem (née Andrews's q-Dyson Conjecture [An]), in a form that would be amenable for the extension promised in the title.

The Károlyi-Nagy Brilliant Proof of Zeilberger-Bressoud

Fact 1: If a polynomial of degree $\leq d$ vanishes at $d+1$ **different** places it must be identically zero.

Proof: By induction on d. If $d = 0$ it is a **constant**, and since it happens to be zero *somewhere*, it must be zero *everywhere*, i.e. it must be the polynomial 0.

If a polynomial $P(x)$ of degree $\leq d$ vanishes at $d+1$ distinct places, let a be one of them. Then (thanks to Euclid), $P(x) = (x-a)Q(x)+c$ for some polynomial $Q(x)$ of degree $\leq d-1$ and constant c, that must be 0, since $P(a) = 0$. Hence $P(x) = (x-a)Q(x)$, and $Q(x)$ is a polynomial of degree $\leq d-1$ that vanishes at d different places. By the induction hypothesis, it is the zero polynomial, and hence so is $P(x)$. □

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Fact 2: (Lagrange Interpolation Formula) If $P(x)$ is a polynomial of degree $\leq d$ in x, and $|A| = d+1$, then

$$P(x) = \sum_{c \in A} \left(\prod_{c' \in A \backslash c} \frac{x - c'}{c - c'} \right) P(c) \quad .$$

Proof: The left-side minus the right side is a polynomial of degree $\leq d$ that vanishes at the $d+1$ different members of A, and hence must be identically zero by Fact 1. □

Fact 3: (Immediate consequence of the Lagrange Interpolation Formula) If $P(x)$ is a polynomial of degree $\leq d$ in x, and $|A| = d+1$, then

$$Coeff_{x^d} P(x) = \sum_{c \in A} \frac{P(c)}{\prod_{c' \in A \backslash c} (c - c')} \quad .$$

Proof: Extract the coefficient of x^d on both sides of Fact 2. □

Fact 4: ('Quantitative' form ([L][KP]) of the Combinatorial Nullstellensatz ([A]), reproved in [KN], Lemma 2.1) Let $F(x_1, \dots, x_n)$ be a polynomial of degree $\leq d_1 + \dots + d_n$. For arbitrary sets $A_1, \dots, A_n$ with $|A_i| = d_i + 1$, the coefficient of $\prod_{i=1}^n x_i^{d_i}$ in $F(x_1, \dots, x_n)$ is

$$\sum_{c_1 \in A_1} \sum_{c_2 \in A_2} \cdots \sum_{c_n \in A_n} \frac{F(c_1, \dots, c_n)}{\phi_1'(c_1) \phi_2'(c_2) \cdots \phi_n'(c_n)} \quad ,$$

where $\phi_i(z) := \prod_{a \in A_i} (z - a)$, and $\phi_i'(c_i)$ is $\phi_i(z)/(z - c_i)$ evaluated at c_i.

New Proof: Since any **poly**nomial of of degree $\leq d_1 + \dots + d_n$ is a linear combination of **mono**mials of degree $\leq d_1 + \dots + d_n$, it suffices, by **linearity**, to prove this for monomials $F = x_1^{m_1} \cdots x_n^{m_n}$ with $m_1 + \dots + m_n \leq d_1 + \dots + d_n$. If there is an i such that $m_i < d_i$ then the left side it 0, and the right side is 0 by Fact 3 with $x = x_i$, $d = d_i$, $P(x_i) = x_i^{m_i}$. If all $m_i \geq d_i$, then, of course, $m_i = d_i$ for all $1 \leq i \leq n$, and the left side is 1 and the right side is $1^n = 1$, by applying Fact 3 for each i, with $P(x_i) = x_i^{d_i}$ and multiplying. □

We are now ready for

Fact 5: (**The Zeilberger-Bressoud Theorem**([ZB])) Let, q and X be commuting indeterminates, and let n be a non-negative integer. Define first

$$(X)_n := \prod_{t=0}^{n-1} (1 - q^t X) \quad .$$

Let $a_1, \dots, a_n$ be non-negative integers, and let $x_1, \dots, x_n$ be commuting indeterminates. The coefficient of $x_1^0 \dots x_n^0$ (i.e. the **constant term**) of

$$\prod_{1 \leq i < j \leq n} (x_i/x_j)_{a_i} (q x_j/x_i)_{a_j} \qquad (qDyson)$$

equals the q-**multinomial coefficient**

$$\frac{(q)_{a_1+a_2+\ldots+a_n}}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}} \quad .$$

Proof ([KN] with purely-routine stuff removed).

If any of the a_i equals 0 then the theorem reduces to one with $< n$ variables and would follow by induction on n, hence we can assume that all the a_i are strictly positive.

Let $\sigma = \sum_{i=1}^{n} a_i$. We have to evaluate the coefficient of $\prod_{i=1}^{n} x_i^{\sigma - a_i}$ of the polynomial (of degree $(n-1)\sigma$)

$$F(x_1, \ldots, x_n) := \prod_{1 \le i < j \le n} (x_i/x_j)_{a_i}(qx_j/x_i)_{a_j} \cdot x_j^{a_i} x_i^{a_j} \quad .$$

Let's Apply Fact 4 with F, $d_i = \sigma - a_i$ and, for $i = 1, \ldots, n$,

$$A_i := \{q^{\alpha_i} \,;\, 0 \le \alpha_i \le \sigma - a_i\} \quad .$$

SubFact 5.1: If there exists a pair i, j with $1 \le i < j \le n$ such that

$$-(a_i - 1) \le \alpha_i - \alpha_j \le a_j \quad ,$$

then $F(q^{\alpha_1}, \ldots, q^{\alpha_n}) = 0$.

Proof: Since $(q^{-e})_f = 0$ if $0 \le e < f$, $(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}$ with $x_i = q^{\alpha_i}$ and $x_j = q^{\alpha_j}$ vanishes, and hence so does F. $\square$

SubFact 5.2: The set of lattice points $(\alpha_1, \ldots, \alpha_n)$ with $0 \le \alpha_i \le \sigma - a_i$ such that for *every* pair $1 \le i < j \le n$ it is not the case that $-(a_i - 1) \le \alpha_i - \alpha_j \le a_j$, in other words the set

$$S(a_1, \ldots, a_n) := \{(\alpha_1, \ldots, \alpha_n) \,;\, 0 \le \alpha_i \le \sigma - a_i\} \bigcap_{1 \le i < j \le n} (\alpha_j - \alpha_i \ge a_i \quad OR \quad \alpha_i - \alpha_j \ge a_j + 1)$$

is the **singleton** set

$$\{(0, a_1, a_1 + a_2, \ldots, a_1 + \ldots + a_{n-1})\} \quad .$$

Proof: The condition, for each $1 \le i < j \le n$

$$\alpha_j - \alpha_i \ge a_i \quad OR \quad \alpha_i - \alpha_j \ge a_j + 1$$

is equivalent to, for each $1 \le i \ne j \le n$,

$$\alpha_j - \alpha_i \ge a_i + [i > j] \quad ,$$

where [*statement*] is 1 and 0 respectively, according to whether the statement is true or false. Obviously all the α_i are **distinct**, hence there exists a unique permutation $\pi \in S_n$ such that

$$\alpha_{\pi(1)} < \alpha_{\pi(2)} < \ldots < \alpha_{\pi(n)} \quad .$$

Because of the conditions, we have

$$\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \geq a_{\pi(i)} + [\pi(i) > \pi(i+1)] \quad .$$

Adding up from $i = 1$ to $i = n-1$ we have

$$\alpha_{\pi(n)} - \alpha_{\pi(1)} \geq \sum_{i=1}^{n-1} a_{\pi(i)} + des(\pi) \quad ,$$

where $des(\pi)$ is the *number of descents* of π (i.e. the number of i, $1 \leq i < n$, for which $\pi(i) > \pi(i+1)$). Hence

$$\alpha_{\pi(n)} - \alpha_{\pi(1)} \geq \sigma - a_{\pi(n)} + des(\pi) \quad .$$

But, $\alpha_{\pi(n)} \leq \sigma - a_{\pi(n)}$ and $\alpha_{\pi(1)} \geq 0$ so, $\alpha_{\pi(n)} - \alpha_{\pi(1)} \leq \sigma - a_{\pi(n)}$, and hence

$$\sigma - a_{\pi(n)} + des(\pi) \leq \sigma - \alpha_{\pi(n)} \quad .$$

Hence $des(\pi) \leq 0$. Of course $des(\pi) \geq 0$, hence $des(\pi) = 0$, and hence π must be the **identity permutation**: $\pi(i) = i$.

Going back to $\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \geq a_{\pi(i)} + [\pi(i) > \pi(i+1)]$ with $\pi = Identity$, we have

$$\alpha_1 \geq 0 \quad , \quad \alpha_2 - \alpha_1 \geq a_1 \quad , \quad \alpha_3 - \alpha_3 \geq a_2 \quad , \quad \ldots \quad , \quad \alpha_n - \alpha_{n-1} \geq a_{n-1} \quad .$$

None of these inequalities can be strict, or else, adding-them-up would imply that $\alpha_n > \sigma - a_n$. Hence, the only solution to the above linear-diophantine system of inequalities is

$$\alpha_1 = 0 \quad , \quad \alpha_2 = a_1 \quad , \quad \alpha_3 = a_1 + a_2 \quad , \quad \ldots \quad , \quad \alpha_n = a_1 + \ldots + a_{n-1} \quad . \quad \Box$$

SubFact 5.3: If $\phi(z) = \prod_{i=0}^{d}(z - q^i)$, and $0 \leq j \leq d$ then $\phi'(q^j) = \prod_{i=0}^{j-1}(q^j - q^i)\prod_{i=j+1}^{d}(q^j - q^i)$ equals $q^{EasyToCompute}(-1)^{AlsoEasyToCompute}$ **times** $(q)_j(q)_{d-j}$. $\Box$

SubFact 5.4: Any evaluation of $F(x_1, \ldots, x_n)$ where $x_i = q^{L_i}$ and L_i are affine-linear expressions in $a_1, \ldots, a_n$ can be written in terms of various $(q)_L$ for some affine-linear expressions L times $q^{EasyToCompute}(-1)^{AlsoEasyToCompute}$ $\Box$

Note: This is better left to a computer, see procedure `EvalFcBG(a,c,r,q,S)` in `qDYSON`.

Plugging-in the unique non-zero point in $S(a_1, \ldots, a_n)$, namely $(0, a_1, \ldots, a_1 + \ldots + a_{n-1})$, and doing purely-routine manipulations (better left to the computer), lo-and-behold, we get what we want, namely the q-multinomial coefficient $\frac{(q)_{a_1+a_2+\ldots+a_n}}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}}$. $\Box$.

(End of proof of Fact 5 (alias the Zeilberger-Bressoud q-Dyson theorem)).

How to Evaluate Any Other Coefficient

Fact 6:(**The Generalized Zeilberger-Bressoud q-Dyson Theorem**) Let $\delta = (\delta_1, \dots, \delta_n)$ be a *fixed*, numeric, vector of integers that add-up-to 0. The coefficient of $\prod_{i=1}^{n} x_i^{\delta_i}$ in the q-Dyson product ($qDyson$) given above is

$$R_\delta(q, q^{a_1}, \dots, q^{a_n}) \cdot \frac{(q)_{a_1+a_2+\dots+a_n}}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}} \quad ,$$

for some easily-computable (using `qDYSON`) *rational function* R_δ. Furthermore, the denominator of R_δ is 'nice' (a product of terms of the form $1-q^L$, where L are affine-linear-combinations in the a_i's).

Proof: *Now* we apply Fact 4 with

$$A_i := \{q^{\alpha_i} \, ; \, 0 \le \alpha_i \le \sigma - a_i + \delta_i\} \quad .$$

SubFact 6.1: If there exists a pair i, j with $1 \le i < j \le n$ such that

$$-(a_i - 1) \le \alpha_i - \alpha_j \le a_j \quad ,$$

then $F(q^{\alpha_1}, \dots, q^{\alpha_n}) = 0$.

Proof: SubFact 6.1 is the same as SubFact 5.1, see the above proof. □

SubFact 6.2: The set of lattice points $(\alpha_1, \dots, \alpha_n)$ with $0 \le \alpha_i \le \sigma - a_i + \delta_i$ such that for *every* pair $1 \le i < j \le n$ it is not the case that $-(a_i - 1) \le \alpha_i - \alpha_j \le a_j$, in other words the set

$$S_\delta(a_1, \dots, a_n) := \{(\alpha_1, \dots, \alpha_n) \, ; \, 0 \le \alpha_i \le \sigma - a_i + \delta_i\} \bigcap_{1 \le i < j \le n} (\alpha_j - \alpha_i \ge a_i \quad OR \quad \alpha_i - \alpha_j \ge a_j + 1)$$

is a **finite** set, easily constructed by the Maple package `qDYSON`, whose cardinality does not depend on $(a_1, \dots, a_n)$, only on δ.

Proof: The condition, for each $1 \le i < j \le n$

$$\alpha_j - \alpha_i \ge a_i \quad OR \quad \alpha_i - \alpha_j \ge a_j + 1$$

is equivalent to, for each $1 \le i \ne j \le n$,

$$\alpha_j - \alpha_i \ge a_i + [i > j] \quad ,$$

where [*statement*] is 1 and 0 respectively, according to whether the statement is true or false. Obviously all the α_i are **distinct**, hence there exists a unique permutation $\pi \in S_n$ such that

$$\alpha_{\pi(1)} < \alpha_{\pi(2)} < \dots < \alpha_{\pi(n)} \quad .$$

Because of the conditions, we have

$$\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \geq a_{\pi(i)} + [\pi(i) > \pi(i+1)] \quad .$$

Adding up from $i = 1$ to $i = n-1$, we have

$$\alpha_{\pi(n)} - \alpha_{\pi(1)} \geq \sum_{i=1}^{n-1} a_{\pi(i)} + des(\pi) \quad ,$$

where $des(\pi)$ is the *number of descents* of π (i.e. the number of i, $1 \leq i < n$, for which $\pi(i) > \pi(i+1)$). Hence

$$\alpha_{\pi(n)} - \alpha_{\pi(1)} \geq \sigma - a_{\pi(n)} + des(\pi) \quad .$$

But, $\alpha_{\pi(n)} \leq \sigma - a_{\pi(n)} + \delta_{\pi(n)}$ and $\alpha_{\pi(1)} \geq 0$ so, $\alpha_{\pi(n)} - \alpha_{\pi(1)} \leq \sigma - a_{\pi(n)} + \delta_{\pi(n)}$, and hence

$$\sigma - a_{\pi(n)} + des(\pi) \leq \sigma - \alpha_{\pi(n)} + \delta_{\pi(n)} \quad .$$

Hence $des(\pi) \leq \delta_{\pi(n)}$. For a given $\pi \in S_n$ that satisfies this condition, we have

$$\sigma - a_{\pi(n)} + des(\pi) \leq \sum_{i=1}^{n-1} (\alpha_{\pi(i+1)} - \alpha_{\pi(i)}) \leq \sigma - a_{\pi(n)} + \delta_{\pi(n)} \quad .$$

Since

$$\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \geq a_{\pi(i)} + [\pi(i) > \pi(i+1)] \quad .$$

we can write, for $i = 1, \ldots, n-1$, and for some integers $m_i \geq 0$ $(1 \leq i \leq n)$: $\alpha_{\pi(1)} = m_1$, and for $1 \leq i \leq n-1$:

$$\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \ = \ a_{\pi(i)} + [\pi(i) > \pi(i+1)] + m_{i+1} \quad .$$

Summing from $i = 1$ to $i = n-1$ gives

$$0 \leq \sum_{i=1}^{n} m_i \leq \delta_{\pi(n)} - des(\pi) \quad .$$

Of course, there are only finitely-many such $\{(m_1, \ldots, m_n)\}$. So, for *each* permutation π obeying $des(\pi) \leq \delta_{\pi(n)}$ and for *each* vector $(m_1, m_2, \ldots, m_n)$ of non-negative integers whose sum is $\leq \delta_{\pi(n)} - des(\pi)$, we have a member of $S_\delta(a_1, \ldots, a_n)$,

$$\alpha_{\pi(i)} = \sum_{r=1}^{i} m_r + \sum_{r=1}^{i-1} (a_{\pi(r)} + [\pi(r) > \pi(r+1)]) \quad .$$

Note that a_i are *symbolic*, and the sets of feasible π and $(m_1, \ldots, m_n)$ *only* depend on δ **not** on $(a_1, \ldots, a_n)$. It is immediately seen that these points satisfy the condition for membership in S_δ. $\square$

SubFact 6.3: For a given feasible permutation π and feasible vector $m = (m_1 \ldots, m_n)$, and with A_i as above, the summand of Fact 4 is a **simple** (factored) rational function of $(q, q^{a_1}, \ldots, q^{a_n})$ **times** the q-multinomial coefficient $\frac{(q)_{a_1+a_2+\ldots+a_n}}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}}$.

Proof: Routine (and programmed into `qDYSON`). $\square$

Adding up these *finitely* many contributions concludes the proof of Fact 6. $\square$

Remark: One can get much smaller sets of evaluation-points S_δ, by shifting the A_i's by (positive or negative) c_i, in other words consider

$$A_i := \{q^{\alpha_i}\,;\, c_i \leq \alpha_i \leq \sigma - a_i + \delta_i + c_i\} \quad .$$

Of course, one should get the *same* output, regardless of the c's, but for the sake of efficiency it would be nice to make S_δ as small as possible. The Maple package **qDYSON** has a procedure `BestShift(d)`, that finds the optimal shift.

Another Remark: Fact 6 is only valid for *numeric* (specific) δ, and each numeric, specific, number of variables n. There is no closed form formula for the general coefficient of $\prod_{i=1}^n x_i^{\delta_i}$ of the q-Dyson product where the δ, as well as $a = (a_1, \dots, a_n)$, are symbolic. For any specific n, it follows from WZ theory[WZ], that this quantity is *holonomic*, i.e. there exist 'pure' linear recurrences (in **each** of a_i) with coefficients that are polynomials in $(q^{a_1}, \dots, q^{a_n}, q^{\delta_1}, \dots, q^{\delta_n})$, but these are already fairly complicated for $n = 3$, and the orders get larger and larger with larger n.

The miracle of q-Dyson is that for the constant term, these recurrences are always *first-order* (that is what it means to be *closed-form*), for **any** number of variables, n, and the generalized Zeilberger-Bressoud (Fact 6) extends this miracle to *any*, specific, other coefficient.

Yet another remark: Even though Fact 6 is only valid, in general, for specific n and specific δ, it sometimes happens that if you take a specific, numeric, $\delta = (\delta_1, \dots, \delta_n)$ then the coefficient of $\delta_r := (\delta_1, \dots, \delta_n, 0(r \;\; times))$ in the q-Dyson product in $n+r$ variables can be expressed 'uniformly', since the size of S_{δ_r} (shifted by a judicious shift) remains the same. Of course, this is the case with the original q-Dyson, with $\delta = (0, \dots, 0)$, where S_δ is always a singleton, leading to a general statement valid for *all* n (i.e. symbolic n). This is also the case with the conjectures of Drew Sills[S2] proved by Lun Lv, Guoce Xin and Yue Zhou [LXZ], by extending the method of [GX]. In [LXZ] there is also a beautiful, much more general theorem. We are sure that their result can be reproved, quicker, using the Károly-Nagy approach, as extended in our present article, but we leave this to the interested reader.

The Maple package qDYSON

Everything (and more) is implemented in the Maple package `qDYSON`. Its only limitation is that the number of variables, n, is *numeric*, not symbolic, but, as noted above, often, by running it for $n \leq 5$, one can deduce, and easily translate for general n, the proofs given by the package for specific n.

The main procedure is: '`Gyula(z,d,q);`'. It inputs a variable-name, z, a numeric list of intgers (of length n, say), $d = [d_1, \dots, d_n]$ whose sum is 0, as well as a variable-name, q. It outputs the rational function $R_d(q, q^{a_1}, \dots, q^{a_n})$ promised by Fact 6, where, for the sake of clarity, q^{a_i} is replaced by z_i. For example, '`Gyula(z,[2,-2,0,0],q);`' yields

$$(-{z_4}^2{z_2}^2q^3z_3z_1+{z_4}^3{z_2}^2q^3z_3z_1+z_3{z_2}^2z_4q^3-{z_2}^2q^3{z_3}^2{z_4}^2z_1+{z_2}^2q^3{z_3}^3{z_4}^2z_1-{z_2}^2z_4q^2z_1z_3-$$

$$z_2q^2z_3-{z_4}^2z_2q^2+z_2z_4q^2z_1z_3-z_2{z_3}^2z_4q^2+z_3z_2z_4q^2-{z_3}^2{z_4}^2z_2q^2z_1+z_2z_4qz_1+z_1z_3z_2q+z_3z_4q-z_1)\cdot$$

$$\frac{(1-z_1)}{(z_4z_2q-1)(z_3z_2q-1)(z_2z_3z_4q-1)(z_3z_2z_4q^2-1)}.$$

As d and/or n gets larger, the size of S_d gets larger, and Maple takes a long time to bring everything under a common denominator (using the command `normal`), so for these, a much faster alternative is the unsimplified version, '`Zoltan (z,d,q);`' , that in some sense is better, since it displays the output as a sum of simple rational functions. For example, '`Zoltan(z,[2,-2,0,0],q);`' yields

$$-\frac{(-1+z_1)(-z_1+q)}{(qz_2z_3z_4-1)(z_3z_2z_4q^2-1)}+\frac{z_2q^2(-1+z_1)(z_4z_1-1)(-1+z_3)(z_2qz_3z_4z_1-1)}{(-1+z_2q)(z_4z_2q-1)(z_2z_3q-1)(z_3z_2z_4q^2-1)}$$

$$+\frac{z_4z_2q^2(-1+z_4)(-1+z_1)(z_3z_1-1)(z_2qz_3z_4z_1-1)}{(z_4z_2q-1)(z_2z_3q-1)(qz_2z_3z_4-1)(z_3z_2z_4q^2-1)}-\frac{q(-1+z_3z_4)(-1+z_1)(z_2z_4qz_1-1)(z_2qz_3z_4z_1-1)}{(-1+z_2q)(z_4z_2q-1)(qz_2z_3z_4-1)(z_3z_2z_4q^2-1)}.$$

Other important procedures are `GyulaTh` , `ZoltanTh`, that are verbose forms of the above, and `Sefer`, that outputs full articles. Some examples are given in the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/qdyson.html` .

Also of interest is procedure '`BestShift(d);`', already mentioned above, that finds the best shift to make S_d as small as possible. There are also plenty of checking procedures, that make sure that everything is correct! Please consult the on-line help (gotten by typing `ezra();`, `ezra1();`, `ezraS();` , and `ezraC();`).

A very brief history

The original Dyson conjecture appeared in 1962, in a very important paper[Dy] (that according to `google scholar` (viewed Aug. 13, 2013) was cited 1483 times), and (whose sequel) lead to, *inter alia*, intriguing connections to the Riemann Zeta function discovered by Hugh Montgomery, and extended by Andrew Odlyzko, Jonathan Keating, Nina Snaith and others. The Dyson conjecture was proved shortly after by Jack Gunson[Gu] and Kenneth G. Wilson(1936-2013, Physics Nobel 1982)[W], who later on went on to revolutionize physics by creating *renormalization group theory.* The *proof from the book* of the original Dyson conjecture (using Fact 3 with $d=n-1$) was given[Go] in 1970 by Bayesian pioneer (and collaborator of Alan Turing at Bletchley Park) Jack Good(1916-2009). In 1982, one of us (Zeilberger) found a longer, but equally nice, *combinatorial proof*[Z]. A couple of years later, Dave Bressoud and Doron Zeilberger succeeded in q-ifing this proof, thereby giving the **first** proof of the q-analog, conjectured, in 1975, by George Andrews[An]. A shorter proof of the Zeilberger-Bressoud q-Dyson theorem was given by Ira Gessel and Guoce Xin [GX], and as already noted, the *proof from the book* was given by Károlyi and Nagy[KN], that formed the inspiration for the present article. Other far-reaching applications of their method are given by Gyula Károlyi in collaboration with Alain Lascoux and Ole Warnaar [KLW].

The problem of computing other coefficients, besides the constant term, for the original Dyson product, was launched by Sills and Zeilberger[SZ], followed by Sills' more general article [S1], that was q-analogized in [S2], where he conjectured interesting 'uniform' expressions for a few other coefficients of the q-Dyson, proved, and vastly generalized, in the article [LXZ], already mentioned above.

Yet another approach

While we *love* the Károlyi-Nagy proof, as extended in this article, let us end by remarking that the *original* approach of [ZB] could also be used to prove Fact 6, and even implemented on a computer. Now the multi-tournaments of [Z] and [ZB] have different score vectors, and unlike the original case, where the 'good guys' can be completely translated into words, with no 'left-overs', now, we do get 'minimal left-overs'. It is easy to see that, for specific δ, and specific n, there are only finitely many of them, and each of them could be q-counted. But, since the Károlyi-Nagy approach is *so* efficient, there is little motivation to extend the original Zeilberger-Bressoud approach.

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