Q-APÉRY IRRATIONALITY PROOFS BY Q-WZ PAIRS

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Abstract. Using WZ pairs, Apéry-style proofs of the irrationality of the q-analogues of the Harmonic series and Ln(2) are given.

0. Introduction:

Let us define the following q-analogues of the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and Ln(2), respectively by:

$$h_q(1) := \sum_{k=1}^{\infty} \frac{1}{q^k - 1} \qquad (\text{for } |q| > 1), \tag{0.1}$$

and

$$Ln_q(2) := \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} \qquad (\text{for } |q| \neq 0, 1). \tag{0.2}$$

In 1948, Paul Erdös [E1] proved the irrationality of $h_2(1)$. Recently, Peter Borwein used Pade approximation techniques [B1] and some complex analysis methods [B2] to prove the irrationality of both $h_q(1)$ and $Ln_q(2)$. Here we present a proof in the spirit of Apéry's magnificent work [A], which was delightfully accounted by Alf van der Poorten [P]. This method of proof gives favorable irrationality measure for $Ln_q(2)$ campared with those implied in [B1], [B2]. Further discussion of irrationality results for certain series is to be found in Erdös [E2].

We will assume familiarity with ref. [Z]. In particular, we use
$\binom{n}{k}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$, where $(q)_n = (1-q)\cdots(1-q^n)$.
N and K are *forward shift operators* on n and k, respectively.
$\Delta_n = N - 1$, $\Delta_k = K - 1$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. A scheme for proving the irrationality of the *q-harmonic series* $h_q(1)$**:**

The claims made in subsections 1.1-1.5 below were found using the Maple Package `qEKHAD` accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

1.1. The q-WZ 1-form ω is:

$$\omega = \frac{-1}{\binom{n+k+1}{k}_q (q)_{n+1}} \{\delta k + \frac{q^{n+1}}{(q^{n+1}-1)} \delta n\}.$$

1.2. The choice of the potential $c(n,k)$ is:

$$c(n,k) = \sum_{m=1}^{n} \frac{q^m}{(1-q^m)(q)_m} + \sum_{m=1}^{k} \frac{1}{(q^m-1)} \frac{1}{\binom{n+m}{m}_q (q)_n}.$$

1.3. The choice of the mollifier $b(n,k)$ is:

$$b(n,k) = (-1)^k q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q.$$

1.4. We define two sequences:

$$a(n) = \sum_{k=0}^{n} c(n,k) b(n,k), \qquad \text{and} \qquad b(n) = \sum_{k=0}^{n} b(n,k).$$

1.5. Introduce $L = y_2(n)N^2 + y_1(n)N + y_0(n)$ and $B(n,k) = P_q^1(n,k)b(n+1,k)$, where

$$A(n,k) = c(n,k)B(n,k) + \frac{(-1)^k q^{2n+3}}{q^{n+1}-1} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}}}{(q)_{n+2}} P_q^2(n,k) \qquad \text{and}$$

$$P_q^1(n,k) = -q\alpha_n^2 \beta_k^{-1}(q^2\alpha_n + 2q) + q\alpha_n^2(q^2\alpha_n^3 + 2q(q+1)\alpha_n^2 + 3q\alpha_n - (q+1) - (\alpha_n+2)\beta_k)$$

$$\begin{aligned} P_q^2(n,k) = q^2\alpha_n^2 + q\alpha_n - 2 + \beta_k(&q^2\alpha_n^5 + q(2q+1)\alpha_n^4 - 2\alpha_n^3 - \alpha_n^3\beta_k - (2-q^{-1})\alpha_n^2\beta_k \\ &- (3q+5)\alpha_n^2 + 2q^{-1}\alpha_n\beta_k + (q-1+2q^{-1})\alpha_n + (1+3q^{-1})), \end{aligned}$$

$$y_0(n) = q(\alpha_n - 1)(q\alpha_n + 2), \;\; y_2(n) = (q\alpha_n - 1)(\alpha_n + 2), \;\; \alpha_n = q^{n+1}, \;\; \beta_k = q^{k+1} \text{ and}$$

$$y_1(n) = q^3\alpha_n^5 + 2q^2(q+1)\alpha_n^4 + q^2\alpha_n^3 - 4q(q+1)\alpha_n^2 + (q^2-4q+1)\alpha_n + 2(q+1).$$

Then

$$(*) \qquad L(b(n,k)) = B(n,k) - B(n,k-1) \qquad \text{and} \qquad L(b(n,k)c(n,k)) = A(n,k) - A(n,k-1).$$

Now, summing over k in $(*)$ shows that both sequences $a(n)$ and $b(n)$ are solutions of $Lu(n) = 0$.

1.6. Set $b_n = b(n)$ and $a_n = a(n)$. Now since $b_{n+1} > b_n$ and $Lb_n = 0$, i.e. $y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0$, then asymptotically we have that

$$\frac{b_{n+2}}{b_{n+1}} = O\left(\frac{y_1(n)}{y_2(n)}\right) = O\left(q^{3n+3}\right).$$

Hence,

$$b_n = O\left(q^{\frac{3n^2}{2}}\right). \tag{1.6.1}$$

On the other hand, $La_n = 0$ and $Lb_n = 0$ lead to the system of recurrence equations,

$$y_2(n)a_{n+2} + y_1(n)a_{n+1} + y_0(n)a_n = 0, \qquad y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0. \tag{1.6.2}$$

Multiplying the first and the second equations in (1.6.2), respectively by b_{n+2} and a_{n+2}, and subtracting we obtain

$$y_1(n)(a_{n+1}b_{n+2} - b_{n+1}a_{n+2}) = y_0(n)(a_{n+2}b_n - b_{n+2}a_n).$$

Rewriting this

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}} = \frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+2}}{b_{n+2}} - \frac{a_n}{b_n}\right).$$

Then we have the estimate

$$\left|\frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}}\right| \le \left|\frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+2}}{b_{n+2}} - \frac{a_{n+1}}{b_{n+1}}\right)\right| + \left|\frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n}\right)\right|.$$

This yields in

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{1.6.3}$$

Therefore,

$$h_q(1) - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{1.6.4}$$

In particular, the sequence of rational numbers $\frac{a_n}{b_n}$ converges moderately faster to $h_q(1)$.

1.7. For a given prime p, let ord_pk denote the exponent of p in the prime expansion of k. Then we observe that

$$ord_p \binom{n}{m}_q \le ord_p(q)_n - ord_p(q)_m. \tag{1.7.1}$$

Note:

$$\binom{n+k}{k}_q \binom{k}{m}_q = \binom{n+m}{m}_q \binom{n+k}{k-m}_q. \tag{1.7.2}$$

Claim: The sequences

$$u_n = a_n(q)_{n+1} \prod_{s=[n/2]}^{n} (1-q^s) \qquad \text{and} \qquad z_n = b_n(q)_{n+1} \prod_{s=[n/2]}^{n} (1-q^s)$$

are polynomials in q with integer coefficients, and moreover

$$z_n = O\left(q^{19n^2/8}\right). \tag{1.7.3}$$

Proof: Applying (1.7.1) and (1.7.2), we estimate the denominator of u_n:

$$\begin{aligned} ord_p \left(\frac{(q^m-1)(q)_n \binom{n+m}{m}_q}{\binom{n+k}{k}_q} \right) &\le ord_p \left(\frac{(q^m-1)(q)_n \binom{k}{m}_q}{\binom{n+k}{k-m}_q} \right) \\ &\le ord_p(q)_n + ord_p(q^m-1) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p(q)_n + ord_p \prod_{s=[n/2]}^{n} (1-q^s) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p \left((q)_n \prod_{s=[n/2]}^{n} (1-q^s) \right), \end{aligned}$$

since $m \le k \le n$. This proves the claim on u_n. And (1.7.3) follows from (1.6.1). The rest is trivial.

Claim: $h_q(1)$ - $\frac{u_n}{z_n} = O\left(\frac{1}{z_n^{1+\delta}}\right)$; where $\delta = 0.26316 \ldots > 0$.

proof: From (1.6.1), (1.6.4) and (1.7.3), we gather that

$$h_q(1) - \frac{u_n}{z_n} = O\left(b_n^{-2}\right) = O\left(q^{-3n^2}\right) = O\left(z_n^{-1-(5/19)}\right).$$

Thus, we have proved:

Theorem 1: If $|q| > 1$ is an integer, $h_q(1)$ is irrational with irrationality measure 4.80.

Note: By invoking Theorem 7 ([Z], p.596) with ω as in 1.1, we obtain the series acceleration:

$$h_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q)_n} \qquad \text{and} \qquad h_q(1) = \sum_{n=1}^{\infty} \frac{1-q^n-q^{2n}}{(q^n-1)\binom{2n}{n}_q (q)_n}.$$

2. A scheme for proving the irrationality of $Ln_q(2)$:

The claims made in subsections 2.1-2.5 below were found using the Maple Package qEKHAD accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

2.1. The qWZ 1-form ω is:

$$\omega = \frac{(-1)^k}{(1-q^{k+1})} \frac{(q)_n}{\binom{n+k+1}{k+1}_q (q^2)_n} \{\delta k + \frac{q^{n+1}}{(1+q^{n+1})}\delta n\}.$$

2.2. The choice of the potential $c(n,k)$ is:

$$c(n,k) = \sum_{m=1}^{n} \frac{q^m (q)_m}{(1-q^m)(q^2)_m} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{(1-q^m)} \frac{(q)_n}{\binom{n+m}{m}_q (q^2)_n}.$$

2.3. The choice of the mollifier $b(n,k)$ is:

$$b(n,k) = q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q.$$

2.4. We define two sequences:

$$a(n) = \sum_{k=0}^{n} c(n,k) b(n,k), \qquad \text{and} \qquad b(n) = \sum_{k=0}^{n} b(n,k).$$

2.5. Introduce $L = y_2(n)N^2 + y_1(n)N + y_0(n)$ and $B(n,k) = P_q^1(n,k) b(n+1,k)$, where

$$A(n,k) = c(n,k)B(n,k) + \frac{(-1)^k q^{2n+3}}{1-q^{n+1}} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}}(q)_{n+1}}{(q^2)_{n+1}} P_q^2(n,k) \qquad \text{and}$$

$$\begin{aligned} P_q^1(n,k) &= q\alpha_n^2 \left[q^3\alpha_n^5 + q^2(1+q)\alpha_n^4 + 2q(1+q^2)\alpha_n^3 - (1-q+q^2)\alpha_n - 3(1+q)\right] \\ &\quad + q\alpha_n^2 \left[q\beta_k^{-1}(q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2-q)\alpha_n - 2) + (q\alpha_n^3 + (q-1)\alpha_n^2 + (2q-1)\alpha_n)\alpha_k - 2\right] \end{aligned}$$

$$\begin{aligned} P_q^2(n,k) &= q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2+q)\alpha_n + 2 - \alpha_n\alpha_k^2\left[\alpha_n^3 + (1-q^{-1})\alpha_n^2 + (2-q)\alpha_n - 2q^{-1}\right] \\ &\quad - \alpha_k\left[q^2\alpha_n^6 + q(1+q)\alpha_n^5 + (2+q+2q^2)\alpha_n^4 + (1+q)\alpha_n^3 + 2\alpha_n^2 - (2+q+q^{-1})\alpha_n + (q^{-1}-1)\right], \end{aligned}$$

$$y_0(n) = -q(\alpha_n - 1)(\alpha_n + 1)(q^2\alpha_n^2 + q\alpha_n + 2), \;\; y_2(n) = -(q\alpha_n - 1)(q\alpha_n + 1)(\alpha_n^2 + \alpha_n + 2),$$

$$y_1(n) = q^4\alpha_n^7 + q^2(1+q)(q\alpha_n^6 + \alpha_n^4) + q(1+q+q^2)(2q\alpha_n^5 + \alpha_n^3) - (1+3q+3q^2+q^3)\alpha_n^2 - (1+q^2)(2+\alpha_n),$$

and $\alpha_n = q^{n+1}$, $\beta_k = q^{k+1}$.

Then

$$(**) \qquad L(b(n,k)) = B(n,k) - B(n,k-1) \qquad \text{and} \qquad L(b(n,k)c(n,k)) = A(n,k) - A(n,k-1).$$

Now, summing over k in $(**)$ shows that both sequences $a(n)$ and $b(n)$ are solutions of $Lu(n) = 0$.

2.6. Similar arguments and estimates as in (1.6) above lead to

$$Ln_q(2) - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{2.6.1}$$

In particular, the sequence of rational numbers $\frac{a_n}{b_n}$ converges moderately faster to $Ln_q(2)$.

2.7. **Claim:** The sequences

$$u_n = a_n \prod_{t=1}^{n}(1+q^t) \prod_{s=[n/2]}^{n}(1-q^s) \qquad \text{and} \qquad z_n = b_n \prod_{t=1}^{n}(1+q^t) \prod_{s=[n/2]}^{n}(1-q^s)$$

are polynomials in q with integer coefficients, and moreover

$$z_n = O\left(q^{19n^2/8}\right). \tag{2.7.1}$$

Proof: Applying (1.7.2) and (2.7.1), we estimate the denominator of u_n:

$$\begin{aligned} ord_p\left(\frac{(1-q^m)(q^2)_n\binom{n+m}{m}_q}{\binom{n+k}{k}_q(q)_n}\right) &\le ord_p\left(\frac{(q^m-1)(q^2)_n\binom{k}{m}_q}{\binom{n+k}{k-m}_q(q)_n}\right) \\ &\le ord_p\left(\frac{(q^2)_n}{(q)_n}\right) + ord_p(q^m-1) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p\left(\frac{(q^2)_n}{(q)_n}\right) + ord_p\prod_{s=[n/2]}^{n}(1-q^s) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p\left(\prod_{t=1}^{n}(1+q^t)\prod_{s=[n/2]}^{n}(1-q^s)\right), \end{aligned}$$

since $m \le k \le n$. This proves the claim on u_n. And (2.7.1) follows from (1.6.1). The rest is trivial.

Claim: $Ln_q(2)$ - $\frac{u_n}{z_n} = O\left(\frac{1}{z_n^{1+\delta}}\right)$; where $\delta = 0.26316\ldots > 0$.

proof: From (1.6.1), (2.6.1) and (2.7.1), we gather that

$$Ln_q(2) - \frac{u_n}{z_n} = O\left(b_n^{-2}\right) = O\left(q^{-3n^2}\right) = O\left(z_n^{-1-(5/19)}\right).$$

Thus, we have proved:

Theorem 2: If $|q| \neq 0, 1$ is an integer, $Ln_q(2)$ is irrational with irrationality measure 4.80.

Note: By invoking Theorem 7 ([Z], p. 596) with ω as in 2.1, we obtain the series acceleration:

$$Ln_q(2) = \sum_{n=1}^{\infty} \frac{q^n(q)_n}{(1-q^n)(q^2)_n} \qquad \text{and} \qquad Ln_q(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(q)_n(1-q^{3n})}{(1-q^n)^2\binom{2n}{n}_q(q^2)_n}.$$

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`The package qEKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

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