Two EZ Proofs of $\sin^2 z + \cos^2 z = 1$

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The two proofs contrast two origins of the sine function: (a)The zeros of $\sin z$, and (b)The power series for $\sin z$.

For the proof based on (a), we study the zeros of $f(z) := \sin^2 z + \cos^2 z - 1$. Observe that f vanishes when $z = n\pi$ or $(n + \frac{1}{2})\pi$, for any integer n. This is the Pythagorean theorem for degenerate triangles. But, even more, f has a second order zero at these points, because f is even (so it must have a double zero at $z = 0$), and periodic of period $\pi/2$.

Now we appeal to a theorem on entire functions. $f(z)$ is an entire function of exponential type 2. In fact $|f(z)| \leq Ce^{2|z|}$. It is a standard consequence of the argument principle that such a function cannot have more than $(2 + \epsilon)r/\pi$ zeros (counting multiplicities) in $|z| \leq r$, unless it vanishes identically. But we have produced $4r/\pi$ zeros there. Thus $f(z) \equiv 0$. $\square$

The same method of proof can be used to prove many identities for elliptic functions.

For the proof based on (b), observe that $\sin z$ (resp. $\cos z$) is the *exponential generating function* (henceforth e.g.f.)[2] for increasing sequences of integers of odd (resp. even) size, weighted by $(-1)^{[size/2]}$. Hence $\sin^2 z + \cos^2 z$ is the e.g.f. for ordered pairs of increasing sequences of integers $(a_1 < \ldots < a_r \,;\, b_1 < \ldots < b_s)$ such that $r + s$ is even, $\{a_1, \ldots, a_r, b_1, \ldots, b_s\} = \{1, 2, \ldots, r + s\}$, and the weight is $(-1)^{[r/2]+[s/2]}$. Let the *mate* of such a pair be $(a_1, \ldots, a_r, b_s \,;\, b_1, \ldots, b_{s-1})$ if $a_r < b_s$, and $(a_1, \ldots, a_{r-1} \,;\, b_1, \ldots, b_s, a_r)$ if $a_r > b_s$. Since every pair has opposite sign from its mate, the total weight of each couple is 0. The only left-over is the pair $(empty, empty)$ that has no mate; Its weight is 1, and its size is 0, hence its $e.g.f.$ is 1. $\square$[3]

The same method of proof can be used to prove many other trig. identities, and also Andre's result that the e.g.f of *up-down permutations* is given by $\sec z + \tan z$.

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[2] The e.g.f. of a combinatorial family of labelled objects according to a weight w is $\sum_{n=0}^{\infty} a_n z^n/n!$, where a_n is the sum of the weights of all the objects of size n. The product A×B of two such combinatorial families is the set of ordered pairs (a,b), a $\in$ A , b$\in$ B, with the labels of a and b disjoint, size(a,b):=size(a)+size(b), and weight(a,b)=weight(a)weight(b). It is easy to see that the e.g.f. of A $\times$ B is the products of the e.g.f.s of A and B. See D.Foata, M.Schutzenberger, LNM # 138 (Springer), and H. Wilf, "generatingfunctionlogy", Academic Press.

[3] A similar (but not identical) proof, that will appear in Math. Magazine, was found independently by Ed Scheinerman.