EXPERIMENTS WITH A POSITIVITY PRESERVING OPERATOR

MANUEL KAUERS * AND DORON ZEILBERGER †

Abstract. We consider some multivariate rational functions which have (or are conjectured to have) only positive coefficients in their series expansion. We consider an operator that preserves positivity of series coefficients, and apply the inverse of this operator to the rational functions. We obtain new rational functions which seem to have only positive coefficients, whose positivity would imply positivity of the original series, and which, in a certain sense, cannot be improved any further.

1. Introduction

Are all the coefficients in the multivariate series expansion about the origin of

$$\frac{1}{1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}$$

positive? Nobody knows. For a similar rational function in three variables, Szegö [7] has shown positivity of the series coefficients using involved arguments. His dissatisfaction with the discrepancy between the simplicity of the statement and the sophistication of the methods he used in his proof has motivated further research about positivity of the series coefficients of multivariate rational functions. For several rational functions, including Szegö's, there are now simple proofs for the positivity of their coefficients available. For others, including the one quoted above [1], the positivity of their coefficients are long-standing and still open conjectures.

In this paper, we consider the positivity problem in connection with the operator T_p ($p \geq 0$) defined as follows:

$$T_p\colon \mathbb{R}[\![x_1,\dots,x_n]\!] \to \mathbb{R}[\![x_1,\dots,x_n]\!],$$
$$(T_p f)(x_1,\dots,x_n) := \frac{f\Bigl(\frac{px_1}{1-(1-p)x_1},\dots,\frac{px_n}{1-(1-p)x_n}\Bigr)}{(1-(1-p)x_1)\cdots(1-(1-p)x_n)}$$

By construction, the operator T_p preserves positive coefficients for any $0 \leq p \leq 1$, i.e., if a power series f has positive coefficients, then the power series $T_p f$ has positive coefficients as well, for any $0 \leq p \leq 1$. For example, via

$$T_{1/2}\Bigl(\frac{1}{1-x-y-z+4xyz}\Bigr) = \frac{1}{1-x-y-z+\frac{3}{4}(xy+xz+yz)},$$

positivity of the former rational function [2] implies positivity of Szegö's rational function [7]. This is a fortunate relation, because the positivity of the former can be shown directly by a simple argument [4] while this is not as easily possible for the latter [5]. (Straub [6] gives a different positivity preserving operator also connecting these two functions.)

This suggests applying the operator T_p "backwards" to a rational function f for which positivity of the coefficients is conjectured, in the hope that this leads to a rational function which again has positive coefficients, and for which positivity of the coefficients is easier to prove. We present some empirical results in this direction. Our results may or may not lead closer to rigorous proofs of some open problems. In either case, we also find them interesting in their own right.

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2. Sharp Improvements

Given a rational function f, we are interested in parameters $p \in [0,1]$ such that $T_p^{-1} f$ has positive series coefficients. Because of $T_p^{-1} = T_{1/p}$, this is equivalent to asking for parameters $p \geq 1$ such that $T_p f$ has positive series coefficients. Clearly, the set of all $p \geq 0$ such that $T_p f$ has positive coefficients forms an interval $[0, p_{\max})$ with a characteristic upper bound $p_{\max}$ for each particular f. Computer experiments have led to the following empirical results.

Empirical Result 1. *Let $f(x,y,z) = 1/(1-x-y-z+4xyz)$. Let p_0 be the real root of $2x^3 - 3x^2 - 1$ with $p_0 \approx 1.68$. Then $p_0 = p_{\max}$.*

Evidence.
1. $p_{\max}$ cannot be larger than p_0, because the particular coefficient $\langle xyz\rangle T_p f = 1 + 3p^2 - 2p^3$ fails to be positive for $p \geq p_0$.
2. CAD computations confirm that all terms $\langle x^n y^m z^k\rangle T_p f$ with $0 \leq n,m,k \leq 50$ are positive for any $0 < p < p_0$.
3. For $p = 2430275/1448618$, all terms $\langle x^n y^m z^k\rangle T_p f$ with $0 \leq n,m,k \leq 100$ are positive. This p is the 15th convergent to p_0 and only about 10^{-14} smaller than this value.
4. For each specific choice of m,k, the terms $\langle x^n y^m z^k\rangle T_p f$ are polynomials in n (and p) of degree $m+k$ with respect to n. For $0 \leq m,k \leq 10$, CAD computations confirm that these are postive for all $n \geq 1$ and all $0 < p < p_0$.

$\square$

Empirical Result 2. *Let $f(x,y,z,w) = 1/(1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw))$. Let p_0 be the real root of $x^4 - 6x^2 - 3$ with $p_0 \approx 2.54$. Then $p_0 = p_{\max}$.*

Evidence.
1. $p_{\max}$ cannot be larger than p_0, because the particular coefficient $\langle xyzw\rangle T_p f = 3 + 6p^2 - p^4$ fails to be positive for $p \geq p_0$.
2. CAD computations confirm that all terms $\langle x^n y^m z^k w^l\rangle T_p f$ with $0 \leq n,m,k,l \leq 25$ are positive for any $0 < p < p_0$.
3. For $p = 730647/287378$, all terms $\langle x^n y^m z^k w^l\rangle T_p f$ with $0 \leq n,m,k,l \leq 240$ are positive. This p is the 15th convergent to p_0 and only about 10^{-12} smaller than this value.
4. For each specific choice of m,k,l, the terms $\langle x^n y^m z^k w^l\rangle T_p f$ are polynomials in n (and p) of degree $m+k+l$ with respect to n. For $0 \leq m,k,l \leq 5$, CAD computations confirm that these polynomial are postive for all $n \geq 1$ and all $0 < p < p_0$.

$\square$

For the rational function f considered in Statement 2, our hope was dispelled that a direct proof for the positivity of $T_{p_{\max}} f$ could be found more easily than for f itself. However, some “suboptimal” values of p do lead to rational functions which have a promising shape. For instance, we found that

$$T_{\sqrt{3}}\Bigl(\frac{1}{1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}\Bigr)$$
$$= \frac{1}{1-x-y-z-w+2(xyz+xyw+xzw+yzw)+4xyzw}.$$

Also note that it seems to be a coincidence that $p_{\max}$ is determined by the coefficient of $xyzw$ in $T_p f$, because this does not hold in the expansion of

$$f(x,y,z,w) = \frac{1}{1-x-y-z-w+\frac{64}{27}(xyz+xyw+xzw+yzw)}$$

which is conjectured to have positive coefficients [5]. Here we have $p_{\max} < 1.66$, by inspection of the coefficients $\langle x^n y^m z^k w^l\rangle T_p f$ for $0 \leq n,m,k,l \leq 100$, while $\langle xyzw\rangle T_p f = -\frac{13}{27}p^4 - \frac{40}{27}p^3 + 6p^2 + 1$ is positive for $p < 2.36$.

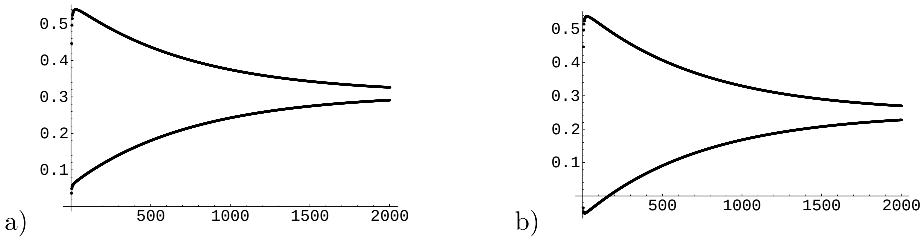

FIGURE 1

3. Asymptotically Positive Coefficients

Inspection of initial coefficients of

$$\frac{1}{1-x-y-z-w+\frac{64}{27}(xyz+xyw+xzw+yzw)}$$

suggests values for $p_{\max}$ that become smaller and smaller as the amount of initial values taken into consideration increases. Is the "real" value $p_{\max}$ determined by the asymptotic behaviour of the coefficients for general p?

Clearly, it is hard to extract conjectures about asyptotic behaviour by just looking at initial values. Instead, such information is better extracted from suitable recurrence equations by looking at the characteristic polynomial and the indicial equation of the recurrence [8]. Using computer algebra, obtaining recurrence equations for the coefficient sequences is an easy task. Often, the asymptotics can be rigorously determined from a recurrence up to a constant multiple K, which cannot be determined exactly, but for which numeric approximations can be found. For instance, if $p > p_0 := (15+3\sqrt{33})/2$, we have

$$\begin{aligned} a_n &:= \langle x^n y^n z^n w^0 \rangle T_p\Big(1/(1-x-y-z-w+\tfrac{64}{27}(xyz+xyw+xzw+yzw))\Big) \\ &\sim K\Big(\frac{(155+27\sqrt{33})\,(-4p+3\sqrt{33}-15)^3}{3456}\Big)^n n^{-1} \quad (n\to\infty) \end{aligned}$$

for $K \gtrsim 0.291$ (Figure 1a shows $a_n/((\cdots)^n n^{-1})$ for $p = p_0 + \frac{1}{10}$, supporting the estimate for K). This is oscillating. For $1 < p < p_0$, the asymptotic behaviour turns into

$$a_n \sim K\big(1+\tfrac{5}{3}p\big)^{3n} n^{-1} \quad (n\to\infty),$$

for $K \gtrsim 0.227$ (Figure 1b shows $a_n/((\cdots)^n n^{-1})$ for $p = p_0 - \frac{1}{10}$, supporting the estimate for K). This is not oscillating, but ultimately positive. This supports the conjecture $p_{\max} < p_0 \approx 16.1168$, which is little news, however, as we already know $p_{\max} < 1.66$ by inspection of initial values. Other paths to infinity that we tried do not give sharper bounds on $p_{\max}$. So it seems that $p_{\max}$ in this example is determined neither by the initial coefficients, nor by the coefficients at infinity, but somehow by the coefficients "in the middle".

We can consider asymptotic positivity of coefficients as an independent question which may also be asked for the rational functions considered in Statements 1 and 2: What are the values $p \geq p_{\max} \geq 1$ such that the series coefficients of $T_p f$ are *ultimately* positive? Denote by $p_{\max}^\infty$ the supremum of these parameters. We have carried out computer experiments in search for $p_{\max}^\infty$, and we obtained the following empirical results.

Empirical Result 3. *Let $f(x,y,z) = 1/(1-x-y-z+4xyz)$. Then $p_{\max}^\infty = 2$.*

Evidence. Let $\epsilon > 0$ (sufficiently small) and $a_{n,m,k} := \langle x^n y^m z^k \rangle T_{2-\epsilon} f$.

(1) First of all, we have $p_{\max}^\infty \leq 2$, because for $\epsilon = 0$, the asymptotics on the main diagonal is

$$a_{n,n,n} \sim K(-27)^n n^{-2/3} \quad (n\to\infty)$$

with $K \gtrsim 0.25$, i.e., $a_{n,n,n}$ is ultimately oscillating for $\epsilon = 0$.

(2) Let $m,k\geq 0$ be fixed and consider $a_{n,m,k}$ as a sequence in n. A direct calculation shows that

$$\begin{aligned} a_{n,m,k} &= \sum_{r=0}^{m}\sum_{t=0}^{k}\sum_{s=0}^{t}(-1)^{r+s}\binom{n}{r}\binom{n+m-r}{m-r}\binom{n+m-2r}{s}\binom{n+m-r+t-s}{t-s} \\ &\qquad\times\binom{r}{k-t}(3-\epsilon)^{r+k-t+s}(3-2\epsilon)^{k-t}(\epsilon-1)^{r-k+t+s} \\ &= \frac{(2-\epsilon)^{2(m+k)}}{m!k!}n^{m+k}+o(n^{m+k})\quad (n\to\infty), \end{aligned}$$

which is positive for $n\to\infty$.

(3) For arbitrary (symbolic) $i\geq 0$ and the particular values $0\leq j\leq 3$, the sequence $a_{n,n+i,j}$ satisfies a recurrence equation of order 3 which gives rise to

$$a_{n,n+i,j}\sim K(3-\epsilon)^{2n}n^{-1/2}\quad (n\to\infty)$$

for some constants K depending on i,j, and ϵ. Numeric computations suggest that these constants are positive, and hence, $a_{n,n+i,j}$ is positive for $n\to\infty$.

(4) For the particular values $0\leq i,j\leq 2$, the sequence $a_{n,n+i,n+j}$ satisfies a recurrence equation of order 3 which gives rise to

$$a_{n,n+i,n+j}\sim K(3-\epsilon)^{3n}n^{-1}\quad (n\to\infty)$$

for some constants K depending on i,j, and ϵ. Numeric computations suggest that these constants are positive, and hence, $a_{n,n+i,n+j}$ is positive for $n\to\infty$.

□

In parts 3 and 4, we could not carry out the arguments for both i and j being generic. We did find a recurrence equation of order 6 for $a_{n,n+i,n+j}$ for generic i,j, with polynomial coefficients of total degree 16 with respect to n,i,j, but this recurrence was way to big for further processing.

Empirical Result 4. *Let $f(x,y,z,w)=1/(1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw))$. Then $p_{\max}^{\infty}=3$.*

Evidence. Let $p\geq 1$ and $a_{n,m,k,l}:=\langle x^n y^m z^k w^l\rangle T_p f$.

(1) First of all, we have $p_{\max}^{\infty}\leq 3$ because for $p>3$, the asymptotics on the main diagonal is determined by the two complex conjugated roots

$$\frac{9+30p^2-7p^4\pm 4p(p^2+3)\sqrt{6-2p^2}}{9}.$$

Their modulus is $(p^2-1)^2$. As $(p^2-1)^2$ itself is not a characteristic root, it follows [3] that $a_{n,n,n,n}$ is ultimately oscillating for $p>3$.

(2) For $i,j,k\geq 0$ fixed, $a_{n,i,j,k}$ is a polynomial in n of degree $i+j+k+1$ whose leading coefficient is $p^{2(i+j+k)}/3^{i+j+k}/i!/j!/k!$. Therefore $a_{n,i,j,k}$ is positive for $n\to\infty$ regardless of p.

(3) For the particular values $i=0,1$ and $0\leq j,k\leq 2$, the sequence $a_{n,n+i,j,k}$ satisfies a recurrence equation of order 3 which gives rise to

$$a_{n,n+i,j,k}\sim K\frac{(p+\sqrt{3})^{2n}}{3^n}n^{j+k-\frac{1}{2}}\quad (n\to\infty)$$

for some constants K depending on i,j,k, and p. Numeric computations suggest that these constants are positive, and hence, $a_{n,n+i,j,k}$ is positive for $n\to\infty$ regardless of p.

(4) For the particular values $0\leq j,k\leq 1$, the sequence $a_{n,n,n+j,k}$ satisfies a recurrence equation of order 4 which gives rise to

$$a_{n,n,n+j,k}\sim K(1+p)^{3n}n^{-1}\quad (n\to\infty)$$

for some constants K depending on j,k, and p. Numeric computations suggest that these constants are positive, and hence, $a_{n,n,n+j,k}$ is positive for $n\to\infty$ regardless of p.

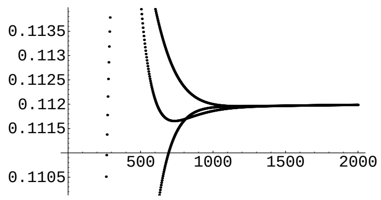

FIGURE 2

(5) The main diagonal $a_{n,n,n,n}$ satisfies a recurrence of order 4 which gives rise to

$$a_{n,n,n,n} \sim K\Big(64 + \tfrac{1}{27}(p^2-9)(2p^4+9p^2+189-2(p^2+3)^{3/2}p)\Big)^n n^{-3/2}$$

for some constant K depending on p.

Numeric computations suggest that K is positive for $p < 3$. For example, Figure 2 shows the quotients $a_{n,n,n,n}/((\cdots)^n n^{-3/2})$ for $p = 3 - \frac{1}{100}$.

$\square$

Stronger evidence in support of the conjectures made in the paper is currently beyond our computational and methodical capabilities. So are rigorous proofs.

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MANUEL KAUERS, RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION, J. KEPLER UNIVERSITY LINZ, AUSTRIA
E-mail address: mkauers@risc.uni-linz.ac.at

DORON ZEILBERGER, RUTGERS UNIVERSITY, PISCATAWAY, NEW JERSEY, USA
E-mail address: zeilberg@math.rutgers.edu