Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory

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Dedicated to Amitai Regev (b. Dec. 7, 1940)

Amitai Regev and Integrals and Sums

Superficially, this article, dedicated with friendship and admiration to Amitai Regev, has nothing to do with either Polynomial Identity Rings, Representation Theory, or Young tableaux, to all of which he made so many outstanding contributions. But anyone who knows even a little about Amitai Regev's remarkable and versatile research, would know that both sums (and multi-sums!), and especially integrals (and multi-integrals!) show up very frequently, e.g. see [R], where one of us (DZ) collaborated in the apppendix that consisted in an explicit evaluation of a certain multi-integral. We should also mention that back in the early eighties, Amitai, together with William Beckner [BR], deduced the then wide-open Macdonald-Mehta conjecture for the classical root systems B-D from Selberg's integral, a fact that was acknowledged in [M] (albeit with characteristic Macdonaldian understatement). Hence, it is clear that sums, multisums, integrals, and multi-integrals, are Amitai's bread and butter, and also cup of tea, so the present work has the potential to help him in his future research.

A Multi-Variable Zeilberger Algorithm

Notation. For k integer, $(z)_k := z(z+1)\dots(z+k-1)$, if $k \geq 0$ and $(z)_k := 1/(z+k)_{-k}$ if $k < 0$. In order to avoid too many subscripts in this article, we will denote $(z)_k$ by $RF(z\,,\,k)$. For any polynomial in $(k_1, \dots, k_r)$ and possibly other variables, $deg(f)$ denotes the **total degree** w.r.t. $(k_1, \dots, k_r)$.

Theorem mZ. Let

$$F(n;k_1,\dots,k_r) = POL(n;k_1,\dots,k_r) \cdot H(n;k_1,\dots,k_r) \quad , \qquad (MultiProperHypergeometric)$$

where $POL(n;k_1,\dots,k_r)$ is a polynomial in $(n,k_1,\dots,k_r)$ and

$$H(n;k_1,\dots,k_r) = \frac{\prod_{j=1}^{A} RF(a''_j\,,\,a'_j n + \sum_{i=1}^{r} a_{ji}k_i)}{\prod_{j=1}^{C} RF(c''_j\,,\,c'_j n + \sum_{i=1}^{r} c_{ji}k_i)} \prod_{i=1}^{r} z_i^{k_i} \quad , \qquad (MultiPureHypergeometric)$$

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The former name (changed Aug. 2005) of M. Apagodu was M. Mohammed.

where the a'_j, c'_j are *non-negative* integers and the a_{ji}, c_{ji} are *integers*, while a''_j, c''_j and $z_1, \dots, z_r$ are *commuting indeterminates*. Then there exists an integer L, to be *explicitly* constructed in the course of the proof, and there exist polynomials in n, $e_0(n), e_1(n), \dots, e_L(n)$, *not all zero*, and there also exist r rational functions of $(n, k_1, \dots, k_r)$, $R_i(n; k_1, \dots, k_r)$ $(i = 1, \dots, r)$, such that

$$G_i(n; k_1, \dots, k_r) := R_i(n; k_1, \dots, k_r) F(n; k_1, \dots, k_r)$$

satisfy

$$\sum_{i=0}^{L} e_i(n) F(n+i, k) = \sum_{i=1}^{r} [G_i(n; k_1, \dots, k_{i-1}, k_i+1, k_{i+1}, \dots, k_r) - G_i(n; k_1, \dots, k_r)] \quad . \quad (WZtuple)$$

Proof. Let

$$\overline{H}(n; k_1, \dots, k_r) = \frac{\prod_{j=1}^{A} RF(a''_j \, , \, a'_j n + \sum_{i=1}^{r} a_{ji} k_i)}{\prod_{j=1}^{C} RF(c''_j \, , \, c'_j (n+L) + \sum_{i=1}^{r} c_{ji} k_i)} \prod_{i=1}^{r} z_i^{k_i} \quad ,$$

and for $i = 1, \dots, r$,

$$f_i(k_1, \dots, k_r) = \prod_{\substack{1 \le j \le A \\ a_{ji} > 0}} RF(a'_j n + a''_j + \sum_{i=1}^{r} a_{ji} k_i \, , \, a_{ji}) \prod_{\substack{1 \le j \le C \\ c_{ji} < 0}} RF(c'_j(n+L) + c''_j + c_{ji} + \sum_{i=1}^{r} c_{ji} k_i \, , \, -c_{ji}) \quad ,$$

and

$$g_i(k_1, \dots, k_r) = \prod_{\substack{1 \le j \le A \\ a_{ji} < 0}} RF(a'_j n + a''_j + a_{ji} + \sum_{i=1}^{r} a_{ji} k_i \, , \, -a_{ji}) \prod_{\substack{1 \le j \le C \\ c_{ji} > 0}} RF(c'_j(n+L) + c''_j + \sum_{i=1}^{r} c_{ji} k_i \, , \, c_{ji}) \quad .$$

Note that

$$\frac{\overline{H}(n; k_1, \dots, k_{i-1}, k_i+1, k_{i+1}, \cdots, k_r)}{\overline{H}(n; k_1, \dots, k_r)} = \frac{f_i(k_1, \dots, k_r)}{g_i(k_1, \dots, k_r)} z_i \quad .$$

Write

$$G_i(n, k) := g_i(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_r) \cdot X_i(k_1, \dots, k_r) \cdot \overline{H}(n; k_1, \dots, k_r) \cdot \quad (Ansatz)$$

Substituting into $(WZtuple)$ and dividing both sides by $\overline{H}(n; k_1, \dots, k_r)$, shows that it is equivalent to

$$\sum_{i=1}^{r} [z_i \cdot f_i(k_1, \dots, k_r) X_i(k_1, \dots, k_{i-1}, k_i+1, k_{i+1}, \dots, k_r) - g_i(k_1, \dots, k_{i-1}, k_i-1, k_{i+1}, \dots, k_r) X_i(k_1, \dots, k_r)]$$

$$- h(k_1, \dots, k_r) = 0 \quad , \qquad (MultiGosper)$$

where

$$h(k_1, \dots, k_r) := \sum_{i=0}^{L} e_i(n) POL(n+i; k_1, \dots, k_r) \cdot \frac{H(n+i; k_1, \dots, k_r)}{\overline{H}(n; k_1, \dots, k_r)} \quad .$$

Note that $h(k_1,\ldots,k_r)$ is a polynomial since

$$\frac{H(n+i;k_1,\ldots,k_r)}{\overline{H}(n;k_1,\ldots,k_r)}=\prod_{j=1}^{A} RF(a''_j+a'_jn+\sum_{i=1}^{r}a_{ji}k_i\,,\,ia'_j)\prod_{j=1}^{C} RF(c''_j+c'_j(n+i)+\sum_{i=1}^{r}c_{ji}k_i\,,\,(L-i)c'_j) \quad .$$

Now write each of the X_i's in generic form, with *undetermined coefficients*, as polynomials in $k_1,\ldots,k_r$ of degree $M := deg(h) - \max(\{deg(f_i), deg(g_i)\})$. Plug them all into $(MultiGosper)$, expand this gigantic polynomial of $k_1,\ldots,k_r$, and equate **all** the coefficients to zero, getting a huge **system of linear homogeneous equations** whose unknowns are the $e_i(n)$'s and the coefficients of the X_i's. There are $r\binom{M+r}{r}+L+1$ unknowns and $\binom{deg(h)+r}{r}$ equations. In order to **guarantee** a *not-all-zero* solution we must insist that #unknowns>#equations. So choose L to be the smallest integer such that

$$r\binom{M+r}{r}+L+1>\binom{deg(h)+r}{r} \quad ,$$

in other words

$$L\geq\binom{deg(h)+r}{r}-r\binom{M+r}{r} \quad .$$

We will now show that such an L exists. Indeed,

$$deg(h)=deg(POL)+L\cdot\max(\sum_{j=1}^{A}a'_j,\sum_{j=1}^{C}c'_j)=b_0+b_1L \quad ,$$

for some specific positive integers b_0, b_1. Also

$$b_2 := max(\{deg(f_i), deg(g_i)\})$$

is some specific positive integer (independent of L). We need to find an L such that

$$r\binom{b_0+b_1L-b_2+r}{r}-\binom{b_0+b_1L+r}{r}+L\geq 0 \quad .$$

For $r=1$ we get that $L=b_2$ will do, and for $r>1$ the left side is a polynomial of L of degree r with a *leading coefficient* that is **positive**, hence tends to ∞ when $L\to\infty$, Hence the inequality holds for sufficiently large L, and the smallest such L is our desired (sharp!) upper bound.

However, so far, we only ruled out the scenario that *all* the $e_i(n)$'s **and** *all* the X_i's are **all** equal to 0. Can it happen that all the $e_i(n)$'s are equal to 0? No way! We are doing things *generically*, in particular with generic z_i's, and if all the $e_i(n)$'s are zero, they would have to be *identically zero* as a function of the z_i's and all the other generic (symbolic) parameters a''_j etc. In particular if we make all the z_i's zero except for a single one, reducing the multi-sum to a *single sum*, then all the $e_i(n)$'s would still have to be identically zero. This scenario has been ruled out in [MZ]. $\square$.

Remark 1. The condition that the a'_j's and c'_j's are *non-negative* integers is w.l.o.g., since one can obtain an equivalent summand with these properties by **shadowing** (see [MZ]).

Remark 2. Theorem mZ is both of theoretical and practical interest. The former because it considerably improves the upper bound for the order of the recurrence established in [WZ]. The latter since it gives an *efficient* algorithm for computing recurrences, superseding the ad-hoc pseudo algorithm that accompanied [WZ].

Remark 3. In *specific* situations (as opposed to the *generic* case), one may be able to get a smaller L, by taking the X_i's to be rational functions, rather than mere polynomials. This is implemented in the Maple package `MultiZeilbergerDen`, where the user is allowed to pick the denominators.

A Multi-Variable q-Zeilberger Algorithm

q-Notation. For k integer, $[a]_k := (1-q^a)(1-q^{a+1})\cdots(1-q^{a+k-1})$, if $k \geq 0$ and $[a]_k := 1/[a+k]_{-k}$ if $k < 0$. In order to avoid too many subscripts, we will denote $[a]_k$ by $qRF(a\,,\,k)$.

Theorem qmZ. Let

$$F(n;k_1,\dots,k_r) = POL(n;k_1,\dots,k_r)\cdot H(n;k_1,\dots,k_r) \quad , \qquad (qMultiProperHypergeometric)$$

where $POL(n;k_1,\dots,k_r)$ is a Laurent polynomial in $(q^n,q^{k_1},\dots,q^{k_r})$, and

$$H(n;k_1,\dots,k_r) = \frac{\prod_{j=1}^{A} qRF(a''_j\,,\,a'_j n + \sum_{i=1}^{r} a_{ji}k_i)}{\prod_{j=1}^{C} qRF(c''_j\,,\,c'_j n + \sum_{i=1}^{r} c_{ji}k_i)}\cdot q^{Q(n;k_1,\dots,k_r)}\cdot\prod_{i=1}^{r} z_i^{k_i} \quad , \qquad (qMultiPureHypergeometric)$$

where the a'_j, c'_j are *non-negative* integers and the a_{ji}, c_{ji} are *integers*, while a''_j, c''_j and $z_1,\dots,z_r$ are *commuting indeterminates*, and $Q(n;k_1,\dots,k_r)$ is a quadratic form in $(n,k_1,\dots,k_r)$. Then there exists an integer L, to be *explicitly* constructed in the course of the proof, and there exist $L+1$ polynomials in q^n, $e_0(q^n), e_1(q^n),\dots,e_L(q^n)$, *not all zero*, and r rational functions of $(q^n,q^{k_1},\dots,q^{k_r})$, $R_i(n;k_1,\dots,k_r)$ $(i=1,\dots,r)$ such that

$$G_i(n;k_1,\dots,k_r) := R_i(n;k_1,\dots,k_r)F(n;k_1,\dots,k_r)$$

satisfy

$$\sum_{i=0}^{L} e_i(q^n)F(n+i,k) = \sum_{i=1}^{r}[G_i(n;k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_r) - G_i(n;k_1,\dots,k_r)] \quad . \qquad (qWZtuple)$$

Plan of Proof: q-analogize the proof of Theorem mZ, in the same way as it was carried out for the single-sum case in [MZ]. □

Remark 4. In many cases one can get a lower order, L, for the recurrence satisfied by the sum $a(n) := \sum_{\mathbf{k}} F(n;\mathbf{k})$, by replacing $F(n;\mathbf{k})$, by its **Paule Symmetrization** [P] (adapted to many variables).

Sharp Upper Bounds for the Almkvist-Zeilberger Algorithm

This section is a discrete-continuous analog of [MZ]. It simplifies (part of) [AZ], and provides a sharp upper bound for the order of the outputted recurrence.

Notation. If f is function of the continuous variable x (among possibly other continuous and/or discrete variables), then $D_x f$ denotes the derivative of f with respect to x, in other words $D_x f := \frac{\partial f}{\partial x}$.

Theorem AZ. Let

$$F(n,x) = POL(n,x) \cdot H(n,x) \quad , \qquad (DiscreteContHypergeometric)$$

where $POL(n,x)$ is a polynomial of (n,x), and

$$H(n,x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^{P} S_p(x)^{\alpha_p} \right) \cdot \left(\frac{s(x)}{t(x)} \right)^n \quad , \qquad (PureDiscreteContHypergeometric)$$

where $a(x), b(x), s(x), t(x)$ and $S_p(x)$ $(1 \leq p \leq P)$ are *polynomials* of x, while the α_p's are commuting *indeterminates.* Let

$$L = deg(b) + deg(s) + deg(t) + \left(\sum_{p=1}^{P} deg(S_p) \right) + max(deg(a), deg(b)) - 1 \quad ,$$

then there exist $L+1$ polynomials in n, $e_0(n), e_1(n), \ldots, e_L(n)$, *not all zero*, and a rational function $R(n,x)$ such that $G(n,x) := R(n,x)F(n,x)$ satisfies

$$\sum_{i=0}^{L} e_i(n) F(n+i, x) = D_x G(n,x) \quad . \qquad (GertDoron)$$

If $F(n,\alpha) = 0$ and $F(n,\beta) = 0$ (and hence $G(n,\alpha) = 0$ and $G(n,\beta) = 0$), it follows, by integrating from $x = \alpha$ to $x = \beta$ that

$$a(n) := \int_{\alpha}^{\beta} F(n,x) dx \quad ,$$

satisfies the linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0 \quad .$$

Proof: Let L, for now, be *any* non-negative integer. Let

$$\overline{H}(n,x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^{P} S_p(x)^{\alpha_p} \right) \cdot \frac{s(x)^n}{t(x)^{n+L}} \quad .$$

We have

$$\sum_{i=0}^{L} e_i(n) F(n+i, x) = h(x) \cdot \overline{H}(n,x) \quad ,$$

where

$$h(x) := \sum_{i=0}^{L} e_i(n) POL(n+i,x) s(x)^i t(x)^{L-i} \quad .$$

Let $q(x)$ and $r(x)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(n,x)$, i.e.

$$\frac{D_x \overline{H}(n,x)}{\overline{H}(n,x)} = \frac{q(x)}{r(x)} \quad .$$

Write

$$G(n,x) = \overline{H}(n,x) \cdot r(x) \cdot X(x) \quad , \tag{$Ansatz$}$$

where $X(x)$ is a polynomial to be determined. Now $(GertDoron)$ is equivalent to

$$(r'(x) + q(x)) \cdot X(x) + r(x) X'(x) = h(x) \quad . \tag{$ContGosper$}$$

Let $M := deg(h) - max(deg(r'+q), deg(r)-1)$, and write $X(x)$ as a polynomial in x of degree M with *undetermined coefficients*. Plugging this into $(ContGosper)$, and equating coefficients, results in $deg(h)+1$ equations for $L+M+2$ unknowns. In order to guarantee a solution, we need

$$L + M + 2 > deg(h) + 1 \quad ,$$

in other words

$$(L+M+2) - (deg(h)+1) \geq 0 \quad ,$$

in other words,

$$L \geq max(deg(r'+q), deg(r)-1) \quad .$$

We leave it to the reader to verify that the expression on the right is indeed

$$L = deg(b) + deg(s) + deg(t) + \left(\sum_{p=1}^{P} deg(S_p) \right) + max(deg(a), deg(b)) - 1 \quad . \square$$

A Multi-Variable Almkvist-Zeilberger Algorithm

The above theorem, and **algorithm**, can be extended to many variables as follows.

Theorem mAZ. Let

$$F(n; x_1, \ldots, x_d) = POL(n; x_1, \ldots, x_d) \cdot H(n; x_1, \ldots, x_d) \quad , \tag{$MultiDiscreteContHypergeometric$}$$

where $POL(n; x_1, \ldots, x_d)$ is a polynomial of $(n, x_1, \ldots, x_d)$, and

$$H(n; x_1, \ldots, x_d) = e^{a(x_1,\ldots,x_d)/b(x_1,\ldots,x_d)} \cdot \left(\prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \right) \cdot \left(\frac{s(x_1, \ldots, x_d)}{t(x_1, \ldots, x_d)} \right)^n \quad , \tag{$PureDiscreteContHypergeometric$}$$

where $a(x_1,\dots,x_d), b(x_1,\dots,x_d)$, $s(x_1,\dots,x_d), t(x_1,\dots,x_d)$ and $S_p(x_1,\dots,x_d)$ $(1 \leq p \leq P)$ are *polynomials* of $(x_1,\dots,x_d)$, while the α_p's are commuting *indeterminates*. There exists a non-negative integer L, to be constructed in the proof, and there exist $L+1$ polynomials in n, $e_0(n), e_1(n), \dots, e_L(n)$, *not all zero*, and there also exist d rational functions $R_i(n;x_1,\dots,x_d)$ $(i=1,\dots,d)$ such that

$$G_i(n;x_1,\dots,x_d) := R_i(n;x_1,\dots,x_d)F(n;x_1,\dots,x_d)$$

satisfy

$$\sum_{i=0}^{L} e_i(n)F(n+i;x_1,\dots,x_d) = \sum_{i=1}^{d} D_{x_i}G_i(n;x_1,\dots,x_d) \quad . \qquad (MultiGertDoron)$$

If $F(n;\pm\infty)=0$ (and hence $G(n;\pm\infty)=0$) it follows, by integrating over $[-\infty,\infty]^d$, that

$$a(n) := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(n;x_1,\dots,x_d)dx_1\dots dx_d \quad ,$$

satisfies the linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^{L} e_i(n)a(n+i) = 0 \quad .$$

Sketch of Proof and Algorithm. Let L, for now, be *any* non-negative integer. Let

$$\overline{H}(n;x_1,\dots,x_d) = e^{a(x_1,\dots,x_d)/b(x_1,\dots,x_d)} \cdot \left(\prod_{p=1}^{P} S_p(x_1,\dots,x_d)^{\alpha_p}\right) \cdot \frac{s(x_1,\dots,x_d)^n}{t(x_1,\dots,x_d)^{n+L}} \quad .$$

We have

$$\sum_{i=0}^{L} e_i(n)F(n+i;x_1,\dots,x_d) = h(x_1,\dots,x_d) \cdot \overline{H}(n;x_1,\dots,x_d) \quad ,$$

where

$$h(x_1,\dots,x_d) := \sum_{i=0}^{L} e_i(n)POL(n+i;x_1,\dots,x_d)s(x_1,\dots,x_d)^i t(x_1,\dots,x_d)^{L-i} \quad .$$

For $i=1,\dots,d$, let $q_i(x_1,\dots,x_d)$ and $r_i(x_1,\dots .x_d)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(n;x_1,\dots,x_d)$ w.r.t. x_i:

$$\frac{D_{x_i}\overline{H}(n;x_1,\dots,x_d)}{\overline{H}(n;x_1,\dots,x_d)} = \frac{q_i(x_1,\dots,x_d)}{r_i(x_1,\dots,x_d)} \quad .$$

Write, for $i=1,\dots,d$,

$$G_i(n;x_1,\dots,x_d) = \overline{H}(n;x_1,\dots,x_d) \cdot r_i(x_1,\dots,x_d) \cdot X_i(x_1,\dots,x_d) \quad , \qquad (Ansatz)$$

where $X_i(x_1, \ldots, x_d)$ are polynomials to be determined. Now $(MultiGertDoron)$ is equivalent to

$$\sum_{i=1}^{d} [D_{x_i} r_i(x_1, \ldots, x_d) + q_i(x_1, \ldots, x_d)] \cdot X_i(x_1, \ldots, x_d) + r_i(x_1, \ldots, x_d) \cdot D_{x_i} X_i(x_1, \ldots, x_d)$$

$$= h(x_1, \ldots, x_d) \quad . \qquad (MultiContGosper)$$

The rest of the proof of the existence of L is analogous to the proof of Theorem mZ, since the degree of h is of the form " **integer+ (positive integer)**$\cdot L$", and for sufficiently large L, the number of unknowns will exceed the number of equations, and we will be guaranteed a solution. Once again, by genericity, it is not possible for all the $e_i(n)$'s to be zero. $\square$

Remark 6. Theorem mAZ sharpens and improves on the work of Akalu Tefera[T], which, in turn, was a great improvement on the pseudo algorithm for multi-integration that accompanied [WZ].

Remark 7. In many cases in practice, one can reduce the order (L), by replacing $(Ansatz)$ by

$$G_i(n; x_1, \ldots, x_d) = \overline{H}(n; x_1, \ldots, x_d) \cdot X_i(x_1, \ldots, x_d) \quad , \qquad (Ansatz')$$

in other words, not assuming that $G_i/\overline{H}$ is divisible by r_i. This is how it is done in `MultiAlmkvistZeilberger`.

Differential Operators

Theorems cAZ and cmAZ below are analogs of AZ and mAZ that treat the case where the integrand's arguments are *all* continuous, including the 'parameter' variable that is not being integrated on. In this case, of course, the output satisfies a linear *differential* equation with polynomial coefficients. Theorems cAZ and cmAZ are not yet implemented in Maple.

Theorem cAZ. Let

$$F(x,y) = POL(x,y) \cdot H(x,y) \qquad (PureContHypergeometric)$$

where $POL(x,y)$ is a polynomial of (x,y), and

$$H(x,y) = e^{\frac{a(x,y)}{b(x,y)}} \cdot \left(\prod_{p=1}^{P} S_p(x,y)^{\alpha_p} \right), \qquad (PureContHypergeometric) \ ,$$

where $a(x,y), b(x,y)$ and $S_p(x,y)$ $(1 \leq p \leq P)$ are polynomials of (x,y) while the α_p are commuting indeterminates.

Let

$$L := deg(b) + \sum_{p=1}^{P} deg(S_p) + max(deg(a), deg(b)) - 1 \ .$$

There exist $L+1$ polynomials, $e_0(x), e_0(x), \ldots, e_L(x)$, not all zero, and rational function $R(x,y)$ such that $G(x,y) := R(x,y)F(x,y)$ satisfies

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x,y) = D_x G(x,y), \quad . \qquad (ContGertDoron)$$

If $F(x,\alpha)=0$ and $F(x,\beta)=0$ (and hence $G(x,\alpha)=0$ and $G(x,\beta)=0$), it follows, by integrating from $y=\alpha$ to $y=\beta$ that

$$a(x) := \int_{\alpha}^{\beta} F(x,y)dy \quad ,$$

satisfies the linear differential equation with polynomial coefficients

$$\sum_{i=0}^{L} e_i(x) D_x^i a(x) = 0 \quad .$$

Sketch of the Proof: The proof makes repeated use of Leibnitz rule together with induction. Let

$$\overline{H}(x,y) = \frac{e^{\frac{a(x,y)}{b(x,y)}}}{b(x,y)^{2L}} \cdot \left(\prod_{p=1}^{P} S_p(x,y)^{\alpha_p - L} \right) \quad .$$

We have

$$\sum_{i=0}^{L} e_i(x) F(x,y) = h(x) \cdot \overline{H}(x,y) \quad ,$$

where $h(x)$ is the expression

$$\sum_{i=0}^{L} a_i(x) \sum_{\substack{k_1+k_2+k_3=i \\ k_i \geq 0}} \binom{i}{k_1,k_2,k_3} (D_x^{k_1} POL(x,y)) \left(\prod_{p=1}^{P} S_p(x,y)^{L-k_2} \right) \cdot T_{k_2}(x,y) \cdot b(x,y)^{2(L-k_3)} \cdot m_{k_3}(x,y) \ ,$$

where $m_{k_3}(x,y)$ is a polynomial in (x,y) for which

$$D_x^{k_2} \left(\prod_{p=1}^{P} S_p(x,y)^{\alpha_p} \right) = \left(\prod_{p=1}^{P} S_p(x,y)^{\alpha_p - k_2} \right) \cdot T_{k_2}(x,y) \ .$$

and $T_{k_3}(x,y)$ is a polynomial in (x,y) such that

$$D_x^{k_3} \left(e^{a(x,y)/b(x,y)} \right) = \frac{m_{k_3}(x,y)}{b(x,y)^{2k_3}} e^{a(x,y)/b(x,y)} \ ,$$

respectively.

Let $q(x)$ and $r(x)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(x,y)$, i.e.

$$\frac{D_x \overline{H}(x,y)}{\overline{H}(x,y)} = \frac{q(x)}{r(x)} \quad .$$

Write

$$G(x,y) = \overline{H}(x,y) \cdot r(x) \cdot X(x) \quad , \qquad (Ansatz)$$

where $X(x)$ is a polynomial to be determined. Now $(ContGertDoron)$ is equivalent to

$$(r'(x) + q(x)) \cdot X(x) + r(x)X'(x) = h(x) \quad . \qquad (ContGosper)$$

Let $M := deg(h) - max(deg(r' + q), deg(r) - 1)$, and write $X(x)$ as a polynomial in x of degree M with *undetermined coefficients*. Plugging this into $(ContGosper)$, and equating coefficients, results in $deg(h) + 1$ equations for $L + M + 2$ unknowns. In order to guarantee a solution, we need

$$L + M + 2 > deg(h) + 1 \quad ,$$

in other words

$$(L + M + 2) - (deg(h) + 1) \geq 0 \quad ,$$

in other words,

$$L \geq max(deg(r' + q), deg(r) - 1) \quad .$$

We leave it to the reader to verify that the expression on the right is indeed

$$L = deg(b) + max(deg(a), deg(b)) + \left(\sum_{p=1}^{P} deg(S_p) \right) - 1 \quad . \square$$

Theorem cmAZ. Let

$$F(x; y_1, \ldots, y_d) = POL(x; y_1, \ldots, y_d) \cdot H(x; y_1, \ldots, y_d) \quad , \qquad (MultiContHypergeometric)$$

where $POL(x; y_1, \ldots, y_d)$ is a polynomial of $(x, y_1, \ldots, y_d)$, and

$$H(x; y_1, \ldots, y_d) = e^{a(x,y_1,\ldots,y_d)/b(x,y_1,\ldots,y_d)} \cdot \left(\prod_{p=1}^{P} S_p(x; y_1, \ldots, y_d)^{\alpha_p} \right) \quad , \qquad (MultiPureContHypergeometric)$$

where $a(y_1, \ldots, y_d), b(y_1, \ldots, y_d)$, $S_p(y_1, \ldots, y_d)$ $(1 \leq p \leq P)$ are *polynomials* of $(y_1, \ldots, y_d)$, while the α_p's are commuting *indeterminates*. There exists a non-negative integer L, to be constructed in the proof, and there exist $L+1$ polynomials in x, $e_0(x), e_1(x), \ldots, e_L(x)$, *not all zero*, and there also exist d rational functions $R_i(x; y_1, \ldots, y_d)$ $(i = 1, \ldots, d)$ such that

$$G_i(x; y_1, \ldots, y_d) := R_i(x; y_1, \ldots, y_d) F(x; y_1, \ldots, y_d)$$

satisfy

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x; y_1, \ldots, y_d) = \sum_{i=1}^{d} D_{x_i} G_i(x; y_1, \ldots, y_d) \quad . \qquad (MultiContGertDoron)$$

If $F(x;\pm\infty)=0$ (and hence $G(x;\pm\infty)=0$) it follows, by integrating over $[-\infty,\infty]^d$, that

$$a(x) := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x;y_1,\dots,y_d)dy_1\dots dy_d \quad ,$$

satisfies the linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^{L} e_i(x) D_x^i a(x) = 0 \quad .$$

Sketch of Proof and Algorithm. Let L, for now, be *any* non-negative integer. Let

$$\overline{H}(x;y_1,\dots,y_d) = \frac{e^{a(x;y_1,\dots,y_d)/b(x;y_1,\dots,y_d)}}{b(x;y_1,\dots,y_d)^{2L}} \cdot \left(\prod_{p=1}^{P} S_p(x;y_1,\dots,y_d)^{\alpha_p}\right) \quad .$$

We have

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x;y_1,\dots,y_d) = h(y_1,\dots,y_d)\cdot \overline{H}(x;y_1,\dots,y_d) \quad ,$$

where $h(y_1,\dots,y_d)$ is the expression given by

$$\sum_{i=0}^{L} a_i(x) \sum_{\substack{k_1+k_2+k_3=i \\ k_i\geq 0}} \binom{i}{k_1,k_2,k_3} (D_x^{k_1} POL(x;y_1,\dots,y_d)) \left(\prod_{p=1}^{P} S_p(x;y_1,\dots,y_d)^{L-k_2}\right) \cdot T_{k_2}(x;y_1,\dots,y_d)\cdot$$

$$b(x;y_1,\dots,y_d)^{2(L-k_3)} \cdot m_{k_3}(x;y_1,\dots,y_d) \quad ,$$

where $T_{k_2}(x;y_1,\dots,y_d)$ is a polynomial in $(y_1,\dots,y_d)$ for which

$$D_x^{k_2}\left(\prod_{p=1}^{P} S_p(x;y_1,\dots,y_d)^{\alpha_p}\right) = \left(\prod_{p=1}^{P} S_p(x;y_1,\dots,y_d)^{\alpha_p-k_2}\right)\cdot T_{k_2}(x;y_1,\dots,y_d) \ .$$

and $m_{k_3}(x;y_1,\dots,y_d)$ is a polynomial in $(x;y_1,\dots,y_d)$ such that

$$D_x^{k_3}\left(e^{a(x;y_1,\dots,y_d)/b(x;y_1,\dots,y_d)}\right) = \frac{m_{k_3}(x;y_1,\dots,y_d)}{b(x;y_1,\dots,y_d)^{2k_3}} e^{a(x;y_1,\dots,y_d)/b(x;y_1,\dots,y_d)} \ ,$$

respectively.

For $i=1,\dots,d$, let $q_i(y_1,\dots,y_d)$ and $r_i(y_1,\dots.y_d)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(x;y_1,\dots,y_d)$ w.r.t. y_i:

$$\frac{D_{y_i}\overline{H}(x;y_1,\dots,y_d)}{\overline{H}(x;y_1,\dots,y_d)} = \frac{q_i(y_1,\dots,y_d)}{r_i(y_1,\dots,y_d)} \quad .$$

Write, for $i = 1, \ldots, d$,

$$G_i(x; y_1, \ldots, y_d) = \overline{H}(x; y_1, \ldots, y_d) \cdot r_i(y_1, \ldots, y_d) \cdot X_i(y_1, \ldots, y_d) \quad , \qquad (Ansatz)$$

where $X_i(y_1, \ldots, y_d)$ are polynomials to be determined. Now $(MultiContGertDoron)$ is equivalent to

$$\sum_{i=1}^{d} [D_{y_i} r_i(y_1, \ldots, y_d) + q_i(y_1, \ldots, y_d)] \cdot X_i(y_1, \ldots, y_d) + r_i(y_1, \ldots, y_d) \cdot D_{y_i} X_i(y_1, \ldots, y_d)$$
$$= h(y_1, \ldots, y_d) \quad . \qquad (MultiContGosper)$$

The rest of the proof of the existence of L is analogous to the proof of Theorem mAZ, since the degree of h is of the form " **integer+ (positive integer)**$\cdot L$", and for sufficiently large L, the number of unknowns will exceed the number of equations, and we will be guaranteed a solution. Once again, by genericity, it is not possible for all the $e_i(x)$'s to be zero. $\Box$

Accompanying Maple packages and Examples

The multi-Zeilberger algorithm, as described in Theorem mZ, is implemented in the Maple package `multiZeilberger`. The refined version, where the user is allowed to specify denominators, is given in `MultiZeilbergerDen`. The q-multi-Zeilberger algorithm, as stated in theorem qmZ, is implemented in the Maple package `qMultiZeilberger`, while the multi-Almkvist-Zeilberger algorithm, as described in Theorem mAZ, is contained in `MultiAlmkvistZeilberger`. Finally `SMAZ` is a more efficient rendition for *symmetric* integrands.

These five packages are available from the webpage of this article `http://www.math.rutgers.edu/~zeilberg/ mamarim/mamarimhtml/multiZ.html`, where there is also sample input and output. Readers can generate many more examples on their own.

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