Automated Counting of Towers (À La Bordelaise)
[Or: Footnote to p. 81 of the Flajolet-Sedgewick Chef-d'œuvre]

Shalosh B. EKHAD and *Doron ZEILBERGER*[1]

À la mémoire de Philippe FLAJOLET

Introduction

One of our favorite theorems in enumerative combinatorics, whose proof-from-the-book by Jean Bétréma and Jean-Guy Penaud[BeP] is succinctly outlined on p. 81 of the Flajolet-Sedgewick[FS] *bible*, (that, in turn, is based on Mireille Bousquet-Mélou's "insightful presentation" [Bo]), is the 3^n theorem[GV] of Dominique Gouyou-Beauchamps and Xavier Viennot.

This amazing result asserts that there are **exactly** 3^n ways of forming a tower of $n+1$ domino pieces such that the bottom floor consists of one or more *contiguous* pieces, and every piece at a higher floor touches one or two pieces on the floor right under it in such a way that the common boundary is exactly (the left- or right-) half of either piece.

For example, here are 100 such towers with 35 pieces:

`http://www.math.rutgers.edu/~zeilberg/viennot/XR1b.html` ,

and here are **all** 27 such towers with four pieces:

`http://www.math.rutgers.edu/~zeilberg/viennot/X4.html` .

We love this proof (and theorem) **so much** that we wrote the admiring *exposé* [Z1].

Why the present article?

Physicists call domino-pieces *dimers*. After writing [Z1], we realized that the same idea still applies to the problem of enumerating towers where instead of using dimers as pieces, one uses trimes, or tetramers, or pentamers, etc. Even more surprisingly, the [BeP] ideas may be used to enumerate towers using *any* given (finite) set of k-mers, where *all* interfaces are allowed. By a k-mer we mean a $1 \times k$ rectangular piece, where the case $k = 2$ is a dimer (alias domino-piece).

This is all implemented in the Maple package `TOWERS`, written by DZ, and linkable from the "front" of this article

[1] Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg at math dot rutgers dot edu` , `http://www.math.rutgers.edu/~zeilberg/` . Written: 12.12.12. Accompanied by the Maple package TOWERS available from `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/migdal.html` , where one can also find lots of output that should be considered part of this article. Supported in part by the NSF.

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/migdal.html` ,

that implements these beautiful human ideas. There you can also find lots of new and exciting enumeration theorems generated by SBE, that readers are welcome to extend even further.

Background

Recall that a domino (alias *dimer*) is a 1×2 rectangular piece. Define an i-mer to be a $1 \times i$ rectangular piece.

Fix a positive integer k, and suppose that we are allowed to use, as pieces, i-mers with $1 \leq i \leq k$. A **tower** is a two-dimensional configuration where the bottom consists of (one or more) **contiguous** pieces (i.e., no gaps). A general tower is formed by starting with a bottom floor (which is already called a tower), and, if desired, building higher floors by placing non-overlapping pieces on top of the currently highest floor, in such a way that every piece in the newly created floor, must touch (at least) one piece of the floor right below it (or else the piece would drop down). Of course, the length of the common boundary between two touching pieces from adjacent floors must be a strictly positive *integer* $\leq k$.

On the computer, we describe a tower as a list of lists where on each floor, we list the locations $[x, x+i]$, of the i-mer whose left-end is at x.

For example, when $k = 3$, the following is one such (legal) tower:

$$[[0,1],[1,2],[2,5],[5,7]],[[3,4],[6,9]],[[3,6]] \quad ,$$

but

$$[[1,2],[2,5],[5,7]],[[3,4],[6,9]],[[4,6]] \quad ,$$

is **not** legal since the piece $[4, 6]$, on the third floor, does not overlap with any piece on the second floor.

Note that it is OK, with our current convention, for a piece to be perfectly aligned with a bottom piece. For example

$$[[0,2],[0,2]] \quad ,$$

is legal (but would not be in the "xaviers" counted by the original 3^n theorem).

Let the *weight* of one piece of size i (i.e. of the form $[a, a+i]$, for some a), be $t^i z_i$, and let the weight of a tower be the *product* of the weights of the individual pieces.

For example,

$$weight\,([[1,2],[2,5],[5,7]],[[3,4],[6,9]],[[3,6]]) = ((tz_1)(t^3z_3)(t^2z_2)) \cdot ((tz_1)(t^3z_3)) \cdot (t^3z_3)) =$$

$$t^{13} z_1^2 z_2 z_3^3 \quad .$$

We are interested in $M_k = M_k(t; z_1, \dots, z_k)$, the generating function (alias weight-enumerator) of the ("infinite") set of *all* legal towers formed from i-mers with $1 \leq i \leq k$. Then the coefficient of t^n would give us the *generating polynomial*, for all towers with area n, and setting all the z_i's to 1 we would get the number of towers of area n.

As in the Bétréma-Penaud approach (see [Z1]), let a *pyramid* be a tower whose first floor only consists of *one* piece, and, let a *half-pyramid* be a pyramid none of whose floors have pieces that lie strictly to the left of the bottom piece.

Let $H = H(t; z_1, \dots, z_k)$, $P = P(t; z_1, \dots, z_k)$, and $M = M(t; z_1, \dots, z_k)$ be the weight-enumerators of half-pyramids, pyramids, and towers, respectively. Then an almost **verbatim** (you do it!) argument shows that H satisfies the following *algebraic equation*:

$$H = \sum_{i=1}^{k} t^i z_i (1+H)^i \quad . \qquad (HalfPyramids)$$

Once we know H, we can get P from:

$$P = \frac{H}{1 - \sum_{i=1}^{k} (i-1) t^i z_i (1+H)^i} \quad , \qquad (Pyramids)$$

and finally

$$M = \frac{P}{1-H} \quad . \qquad (Towers)$$

If the set of piece-sizes is not $\{1, \dots, k\}$, but an arbitrary finite set of positive integers, S, then of course

$$H = \sum_{i \in S} t^i z_i (1+H)^i \quad .$$

From H we can get P:

$$P = \frac{H}{1 - \sum_{i \in S} (i-1) t^i z_i (1+H)^i} \quad ,$$

and finally

$$M = \frac{P}{1-H} \quad .$$

If we only want straight-enumeration, then set all $z_i = 1$, getting

$$H = \sum_{i \in S} t^i (1+H)^i \quad . \qquad (HalfPyramids1)$$

From this we can get P:

$$P = \frac{H}{1 - \sum_{i \in S} (i-1) t^i (1+H)^i} \quad , \qquad (Pyramids1)$$

and finally

$$M = \frac{P}{1-H} \quad . \qquad (Towers1)$$

It is very easy, for a computer, by iterating $H \to \sum_{i \in S} t^i (1+H)^i$, starting with $H=0$, to crank out the first 200 terms, but it gets slower and slower for higher terms. But thanks to Comtet's famous theorem, implemented in the Salvy-Zimmermann Maple package `gfun` [SaZ], we can (rigorously) find a linear differential equation with polynomial coefficients, and from this (still using `gfun`) a linear recurrence equation for the enumerating sequence of half-pyramids, that would give you, *much faster* than the algebraic equation, the 50000-th term, say.

Also from $(HalfPyramids1)$, the algebraic equation for H, one can derive algebraic equations for P and M using $(Pyramids1)$ and $(Towers1)$ respectively.

Furthermore, from these algebraic equations, one can use the beautiful methods described in [FS] to derive asymptotics. Alternatively, one can derive them from the linear recurrences by using the Maple package `AsyRec` described in [Z2].

But an even more efficient approach is an **empirical** one. First use the algebraic equations to crank out the first few terms of the desired sequences, *guess* linear recurrences, then use them to crank out **many** terms, and use empirical asymptotics to estimate the asymptotics. So everything is, if not "rigorous", at least semi-rigorous, and easily *rigorizable*. Being *empiricists*, we prefer the latter method.

Encore: The One Piece-Size Case

I. All Interfaces are allowed

If there is only one piece, of size k, and *all interfaces are allowed* then the generating function, H, in the variable b, for the number of half-pyramids with n pieces (so $b = t^k$) is:

$$H = b(1+H)^k \quad ,$$

and by *Lagrange Inversion*, we get the humanly-generated fact that the number of half-pyramids (where all interfaces are allowed) using n pieces, each of length k, equals $\frac{(kn)!}{n!(kn-n+1)!}$, and thanks to $P = \frac{H}{1-(k-1)H}$, we get that the number of such pyramids is $\frac{1}{k}\binom{kn}{n}$. For $k>2$, there is no 'nice' expression for the number of towers with n pieces, but of course, using $M = \frac{H}{(1-(k-1)H))(1-H)}$, one can get a linear recurrence, either directly, or better still, empirically.

For $k=2$ we have a **nice surprise**, the number of domino towers with n pieces, where all interfaces are allowed, is 4^{n-1}, in nice analogy with the fact that the number of xaviers (where the exact-alignment interface is forbidden) is 3^{n-1}.

II. All Interfaces are allowed, except Exact Alignment

Recall that in the original, Viennot, scenario, that lead to the beautiful 3^{n-1} formula, with dimers (dominoes), it was forbidden to place a piece exactly aligned with a piece on the floor below it. If we impose this rule for the *one-piece case*, of size k, (unfortunately, the analysis is inapplicable for structures with more than one piece-size), we get the equation

$$H = b((1+H)^k - H) \quad ,$$

with, as above $P = \frac{H}{1-(k-1)H}$, $M = \frac{H}{(1-(k-1)H))(1-H)}$.

For $k > 2$, things are no longer closed-form, but we still get linear recurrences with polynomial coefficients, since the generating functions are algebraic, and hence D-finite.

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