A Short Proof of a Ptolemy-Like Relation for an Even number of Points on a Circle Discovered by Jane McDougall

Marc CHAMBERLAND and Doron ZEILBERGER

In fond memory of Andrei Zelevinsky (1953-2013) who loved Ptolemy's theorem

Jane McDougall[M] has discovered (and apparently found a complicated proof using heavy machinery) the following beautiful Ptolemy-style theorem relating the distances amongst any even number of points on a circle.

McDougall's Theorem: Let n be a positive integer, and let P_i $(1 \leq i \leq 2n)$ be points on a circle. Let $d_{i,j}$ be the distance between P_i and P_j, and let

$$R_i := \prod_{\substack{1 \leq j \leq 2n \\ j \neq i}} d_{i,j} \quad .$$

Then

$$\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}} \quad .$$

Proof: Without loss of generality the circle is the unit circle. Let

$$P_i = (\cos 2t_i \, , \, \sin 2t_i) \quad (1 \leq i \leq 2n) \quad .$$

Thanks to trig, $d_{i,j} = 2\sin(t_j - t_i)$, and thanks to DeMoivre, this equals

$$-\sqrt{-1}\left(e^{\sqrt{-1}(t_j-t_i)} - e^{-\sqrt{-1}(t_j-t_i)}\right) \quad .$$

Let

$$u_i = e^{\sqrt{-1}t_i} \quad .$$

It follows that

$$d_{i,j} = -\sqrt{-1}\left(\frac{u_j}{u_i} - \frac{u_i}{u_j}\right) = -\sqrt{-1}\frac{u_j^2 - u_i^2}{u_i u_j} \quad (i < j) \quad .$$

Hence

$$(-\sqrt{-1})^{2n-1}(-1)^{i-1}\frac{1}{R_i} = (\prod_{j=1}^{2n} u_j)\frac{u_i^{2n-2}}{\prod_{j \neq i}(u_j^2 - u_i^2)} \quad .$$

So we have

$$(\sqrt{-1})^{2n-1}\sum_{i=1}^{2n}\frac{(-1)^i}{R_i} = \left(\prod_{j=1}^{2n} u_j\right)\sum_{i=1}^{2n}\frac{u_i^{2n-2}}{\prod_{j \neq i}(u_j^2 - u_i^2)} \quad . \qquad (Jane)$$

Recall the *Lagrange Interpolation Formula*: If $P(z)$ is a polynomial of degree $\leq N-1$ in z then, for any distinct numbers $z_1, \dots, z_N$,

$$P(z) = \sum_{i=1}^{N}\frac{(z-z_1)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_N)}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)}P(z_i) \quad . \qquad (Joseph)$$

(Let's recall the trivial proof: both sides are polynomials of degree $\leq N-1$ that coincide at the N values $z=z_1,\ldots,z=z_N$, so they must be identically equal.)

Taking the polynomial to be $P(z)=z^r$ (with $r<N-1$), and equating the coefficient of z^{N-1} on both sides of $(Joseph)$, gives the identity:

$$0=\sum_{i=1}^{N} \frac{z_i^r}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)} \quad . \qquad (Joseph')$$

Now take $N=2n$, $r=n-1$, and $z_i=u_i^2$ in $(Joseph')$ and conclude that the right side of $(Jane)$ is indeed zero. $\square$

It follows that if our $2n$ points lie on a line (the case $R=\infty$), the theorem is true as well. Furthermore, in that case it is *even* true for an **odd** number of points. We leave this as an exercise to the dear readers.

Reference

[M] Jane McDougall, *private communication*:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/JaneMcDougallMessage.html`
.

Marc Chamberland, Department of Mathematics and Statistics, Grinnell College, Grinnell, Iowa 50112, USA. `chamberland at grinnell dot edu`

Doron Zeilberger, Mathematics Department, Rutgers University (New Brunswick), Piscataway, NJ 08854, USA. `zeilberg at math dot rutgers dot edu`

April 15, 2013