GRAPHICAL MAJOR INDICES

Dominique FOATA[1] *and Doron ZEILBERGER*[2]

Abstract: A generalization of the classical statistics "maj" and "inv" (the major index and number of inversions) on words is introduced, parameterized by arbitrary graphs on the underlying alphabet. The question of characterizing those graphs that lead to equi-distributed "inv" and "maj" is posed and answered.

Résumé: On introduit une généralisation des statistiques classiques que sont "maj" et "inv" (l'indice majeur et le nombre d'inversions) sur les mots, qui est paramétrisée par des graphes arbitraires sur l'alphabet sous-jacent. La question de caractériser ces graphes conduisant à des statistiques "inv" et "maj" qui soient équidistribuées est posée et résolue.

0. Introduction

Every mathematician knows what the *the number of inversions* of a permutation is, as it features in the definition of the determinant. The number of inversions of a permutation of length n,

$$\operatorname{inv}\pi = \sum_{1\le i\le j\le n} \chi(\pi(i) > \pi(j)),$$

(using the classical notation $\chi(A) = 1$ or 0, depending on whether the statement A is true of false) is a measure of how 'scrambled' it is compared to the identity permutation $[1, 2, \ldots, n]$. Netto proved (and it is nowadays easy to see, e.g., [Kn73, p. 15]) that the generating function for "the number of inversions"

$$\sum_{\pi\in S_n} q^{\operatorname{inv}(\pi)},$$

equals the q-analog of $n!$, i.e., $[n]! := 1(1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1})$, that, can be also written as $(q)_n/(1-q)^n$, where, as usual in $q-$theory, $(q)_n = (1-q)(1-q^2)\ldots(1-q^n)$.

---

[1] Département de mathématique, Université Louis Pasteur, 7, rue René-Descartes, F-67084 Strasbourg, France. (`foata@math.u-strasbg.fr`). Supported in part by the E.E.C. program on Algebraic Combinatorics.

[2] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. (`zeilberg@math.temple.edu`). Supported in part by the NSF.

The number of inversions is an example of a *permutation statistic*, by which is meant a numerical attribute that permutations possess (just like height, weight, or number of children for humans). The utility of the generating function according to a given statistic "stat,"

$$F_{\mathrm{stat}}(q) := \sum_{\pi \in S_n} q^{\mathrm{stat}(\pi)},$$

is that it contains in it all the 'statistical' information regarding "stat." Also its derivatives evaluated at $q = 1$ enable us to, successively, find the average, standard deviation, and higher moments of its distribution. Furthermore, when the generating function is 'nice' it hints at (combinatorial, algebraic and sometimes analytic) structures.

MacMahon [Mac15, p. 135] was the first to introduce another such statistic, that he called 'the greater index', but that is nowadays called the 'major index' and denoted by "maj." In fact, he defined that statistic not only for permutations but for arbitrary *words* with possible repetitions of letters. He did also the same for "inv." If X is a totally ordered alphabet, and if $w = x_1 x_2 \ldots x_m$ is a word with letters in X, those two statistics are defined by

$$\begin{aligned} \operatorname{maj} w &= \sum_{i=1}^{m-1} i\, \chi(x_i > x_{i+1}), \\ \operatorname{inv} w &= \sum_{1 \le i < j \le m} \chi(x_i > x_j). \end{aligned}$$

To restate MacMahon's result we will take the alphabet X as the linear set $[\,r\,] = \{1, 2, \ldots, r\}$ $(r \ge 1)$. Let $\mathbf{c} = (c(1), c(2), \ldots, c(r))$ be a sequence of r non-negative integers and let v be the (non-decreasing) word $v = 1^{c(1)} 2^{c(2)} \ldots r^{c(r)}$. We will denote by $R(v)$ (or by $R(\mathbf{c})$ if there is no ambiguity) the class of all rearrangements of the word v, i.e., the class of all words containing exactly $c(i)$ occurrences of the letter i for all $i = 1, \ldots, r$. Then MacMahon [Mac13] (see also [Mac78]) proved that for each integer k there are as many words $w \in R(\mathbf{c})$ such that $\operatorname{maj} w = k$, as there are words $w' \in R(\mathbf{c})$ such that $\operatorname{inv} w' = k$. In other words, the statistics "maj" and "inv" are equidistributed on each rearrangement class.

It is well known, and easy to see, that the number of words in $R(\mathbf{c})$ is the multinomial coefficient:

$$\binom{c(1) + c(2) + \cdots + c(r)}{c(1), c(2), \ldots, c(r)} = \frac{(c(1) + c(2) + \cdots + c(r))!}{c(1)!\, c(2)! \ldots c(r)!}.$$

MacMahon's proof [Mac13, Mac78] (see also [Kn73, p. 17], [An76, chap. 3]) of the forementioned result was to show that the generating functions for "inv" and "maj", over the class $R(v)$, i.e., $\sum_w q^{\mathrm{inv}\, w}$ and $\sum_w q^{\mathrm{maj}\, w}$ (with w runnning over the class $R(\mathbf{c})$), were *both* given by the q-analog of the multinomial coefficient:

$$\begin{bmatrix} c(1)+c(2)+\cdots+c(r) \\ c(1), c(2), \ldots, c(r) \end{bmatrix} = \frac{(q)_{c(1)+c(2)+\cdots+c(r)}}{(q)_{c(1}(q)(_{c(2)}\cdots(q)(^{c(r)}}.$$

The natural question of finding a bijection that sends each permutation to another one in such a way that the major index of the image equals the number of inversions of the original, has been answered by the first author [Fo68], and since 'canonized ' in *the book* ([Kn73], ex. 5.1.1.19).

In this paper we introduce a natural generalization of both "inv" and "maj," parameterized by a general directed graph. A *directed graph* on X is any subset U of the Cartesian product $X \times X = \{(x,y) \mid 1 \le x \le r, 1 \le y \le r\}$. Of course there are altogether 2^{r^2} directed graphs.

For each such directed graph U let's associate the following statistics defined on each word $w = x_1 x_2 \ldots x_m$ by

$$(0.1) \qquad \begin{aligned} \mathrm{maj}'_U\, w &= \sum_{i=1}^{m-1} i\, \chi\big((x_i, x_{i+1}) \in U\big), \\ \mathrm{inv}'_U\, w &= \sum_{1 \le i < j \le m} \chi\big((x_i, x_{i+1}) \in U\big). \end{aligned}$$

Further in the paper other statistics "maj_U" and "inv_U" (without any primes) will be introduced.

The purpose of this paper is to characterize the directed graphs U that posses the 'Mahonian property' of "inv" and "maj" having the same generating function. We first need the following definition.

Definition. An *ordered bipartition* of X is a sequence $(B_1, B_2, \ldots, B_k)$ of non-empty disjoint subsets of X, of union X, together with a sequence $(\beta_1, \beta_2, \ldots, \beta_k)$ of elements equal to 0 or 1. If $\beta_l = 1$ (resp. 0), we say that the subset B_l is *underlined* (resp. *non-underlined*). For the sake of convenience, we also say that the *subscript* l or *each element* of B_l is *underlined* (resp. *non-underlined*).

A relation U on $X \times X$ is said to be *bipartitional*, if there exists an ordered bipartition $((B_1, B_2, \ldots, B_k),\ (\beta_1, \beta_2, \ldots, \beta_k))$ such that $(x,y) \in U$ iff either $x \in B_l$, $y \in B_{l'}$ and $l < l'$, i.e., if the block containing x is to the left of the block containing y, or x and y belong to the same block B_l and B_l is underlined.

As proved by Han [Han95], a bipartitional relation U can also be characterized by the following two relations

$$\begin{aligned} &(x,y) \in U,\ (y,z) \in U \Rightarrow (x,z) \in U\,; \\ &(x,y) \notin U,\ (z,y) \in U \Rightarrow (z,x) \in U. \end{aligned}$$

Some particular bipartitional relations are worth being noticed.

1) $U = \{(x,y) \mid x > y\}$ that corresponds to the ordered bipartition $(\{r\},\ldots,\{2\},\{1\})$; in this case $\mathrm{inv}_U = \mathrm{inv}$ and $\mathrm{maj}_U = \mathrm{maj}$;

2) $U = \{(x,y) \mid x \ge y\}$ that is associated with the ordered bipartition $(\underline{\{r\}},\ldots,\underline{\{2\}},\underline{\{1\}})$, where all the blocks are underlined; the inversions and descents involved in the statistics "inv_U" and "maj_U" also include all the pairs (x,x);

3) $U = \emptyset$ which is associated with the one-block ordered bipartition $(\{1,2,\ldots,r\})$; the statistics "inv_U" and "maj_U" are identically zero;

4) $U = X \times X$ which is associated with the one-underlined-block ordered bipartition $(\underline{\{1,2,\ldots,r\}})$; in this case $\mathrm{inv}_U\, w = \mathrm{maj}_U\, w = m(m-1)/2$ for each word w of length m;

5) U which is associated with an ordered bipartition all the blocks of which are singletons; such relations have been considered by Clarke and Foata [ClFo94, ClFo95a, ClFo95b] who also introduced the statistic "maj_k" which is immediately related with the statistic "maj_U" further defined.

A bipartitional relation $U = ((B_1, B_2, \ldots, B_k),\ (\beta_1, \beta_2, \ldots, \beta_k))$ can also be visualized as follows: rearrange the elements of X in a row in such a way that the elements of B_1 come first, in any order, then the elements of B_2, etc. Then U will consist of all the block products $B_l \times B_{l'}$ with $l < l'$, as well as the block product $B_l \times B_l$ whenever B_l is underlined.

In Figure 1, for instance, the underlying ordered bipartition consists of four blocks (B_1, B_2, B_3, B_4) with B_1, B_4 underlined.

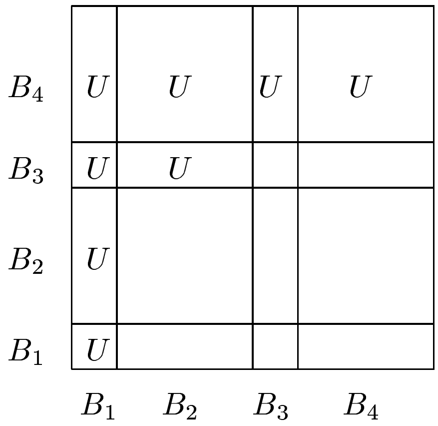

Fig. 1

Our first result is the following.

Theorem 1. *The statistics "inv'_U" and "maj'_U" are equidistributed on each rearrangement class, if and only if the relation U is bipartitional.*

We first prove the 'easy' part, which as usual is the 'if' part. Three proofs will be given. The first manipulative, the second combinatorial à la MacMahon, the third bijective, as people say to-day. All this is derived in sections 3, 4 and 5, respectively. Section 6 contains the proof of the 'only if' part.

Now if U is a bipartitional relation on X, two other statistics "inv_U" and "maj_U" may be defined, that also reduce to "inv" and "maj" when $U = (\{r\}, \ldots, \{2\}, \{1\})$. Let $|w|_-$ denote the number of underlined letters in the word $w = x_1 x_2 \ldots x_m$. Then define

$$(0.2)\qquad \begin{aligned} \mathrm{maj}_U\, w &= \sum_{i=1}^{m-1} i\,\chi\big((x_i, x_{i+1}) \in U\big)\big) + m\,\chi(x_m \text{ is underlined}),\\ \mathrm{inv}_U\, w &= \sum_{1\le i<j\le m} \chi\big((x_i, x_j) \in U\big)\big) + |w|_- \,. \end{aligned}$$

In other words, maj_U is equal to $\mathrm{maj}'_U\, w$ plus the *length* of the word whenever the last letter is underlined, while $\mathrm{inv}_U\, w$ is equal to $\mathrm{inv}'_U\, w$ plus the *number of underlined letters* in w.

We say that a bipartitional relation U is *compatible*, if all its underlined blocks are *on the left* of its non-underlined ones, or, with the above notations, if the sequence $(\beta_1, \beta_2, \ldots, \beta_k)$ is of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$.

We next prove the theorem.

Theorem 2. *Let U be a bipartitional relation on X. Then "maj_U" and "inv_U" are equidistributed on each rearrangement class, if and only if U is compatible.*

As we shall see, the notion of compatibility is crucial. It relates with an analogous notion introduced in Clarke and Foata (*op. cit.*) for dealing with the number of *excedances* and the *Denert statistic*. If U is non-compatible, "maj_U" and "inv_U" are not even equidistributed on a class of two elements. For example, let $X = \{1, 2\}$ and let U be the (non-compatible) bipartitional relation associated with the ordered bipartition $(\{1\}, \underline{\{2\}})$. Then $\mathrm{inv}_U\, 1, 2 = 2$, $\mathrm{inv}_U\, 2, 1 = 1$, while $\mathrm{maj}_U\, 1, 2 = 3$, $\mathrm{maj}_U\, 2, 1 = 0$. Actually, that simple example is the core of the proof of the 'only if' part of Theorem 2 (see section 7).

Let U be an ordered bipartition. Parallel to the definition of "maj'_U" and "maj_U" we can also introduce two kinds of *U-descents*. Let $w = x_1 x_2 \ldots x_m$ be a word; we say that there is a *U-descent of the first kind* at i in w, if $1 \le i \le m-1$ and $(x, x_{i+1}) \in U$, and a *U-descent of the second kind* at i in w, if $1 \le i \le m-1$ and $(x, x_{i+1}) \in U$ or $i = m$ and x_m is underlined. Denote by $\mathrm{des}'_U\, w$ (resp. $\mathrm{des}_U\, w$) the number of those U-descents of the first kind (resp. of the second kind). In section 4 we derive an expression for the generating function for each rearrangement class $R(\mathbf{c})$ by the pair of statistics $(\mathrm{des}'_U, \mathrm{maj}'_U)$.

Section 7 contains the calculation of the generating function of $R(\mathbf{c})$ by the pair $(\mathrm{des}_U, \mathrm{maj}_U)$ and also the proof of Theorem 2. A bijective proof of the latter Theorem appears in section 8.

1. Enumerating bipartitional relations

For each $r \ge 1$ let b'_r (resp. b_r) be the number of bipartitional relations (resp. compatible bipartitional relations) on a set of cardinality r. Also let $b'_0 = b_0 = 1$. The exponential generating functions for both sequences (b'_r) and (b_r) are easily derived and, using MAPLE, their first values calculated. Denote by $S(r,k)$ $(1 \le r \le k)$ the sequence of the Stirling numbers of the second kind (see, e.g., [Co70, vol. 2, p. 40]).

Proposition 1.1. *We have the formulas*

$$b'_r = \sum_{k=1}^{r} S(r,k) k!\, 2^k\,; \tag{1.1}$$

$$b_r = \sum_{k=1}^{r} S(r,k)\, (k+1)!\,; \tag{1.2}$$

$$\sum_{r\ge 0} b'_r \frac{u^r}{r!} = \frac{1}{3-2e^u} \tag{1.3}$$
$$= 1 + 2u + 10\frac{u^2}{2!} + 74\frac{u^3}{3!} + 730\frac{u^4}{4!} + 9002\frac{u^5}{5!} + 133210\frac{u^6}{6!} + \cdots$$

$$\sum_{r\ge 0} b_r \frac{u_r}{r!} = \frac{1}{(2-e^u)^2} \tag{1.4}$$
$$= 1 + 2u + 8\frac{u^2}{2!} + 66\frac{u^3}{3!} + 308\frac{u^4}{4!} + 2612\frac{u^5}{5!} + 25988\frac{u^6}{6!} + \cdots$$

Proof. Formulas (1.1) and (1.2) follow immediately from the combinatorial definition of the Stirling numbers. Accordingly, we can easily derive (1.3) and (1.4) from the "vertical" exponential generating function for the Stirling numbers. A more direct and conceptual proof consists of making use of the partitional complex approach [Fo74] (or invoking the theory of species dear to our québécois friends [Be94]). This goes as follows.

Suppose that for each $r \ge 1$ there are two blocks of size r, say, the underlined $\underline{[\,r\,]}$ and the non-underlined block $[\,r\,]$. The exponential generating function for those two kinds of blocks is

$$G = 2\frac{u^1}{1!} + 2\frac{u^2}{2!} + \cdots + 2\frac{u^n}{n!} + \cdots = 2e^u - 2.$$

Hence the expansion of $(1-G)^{-1}$ will be the generating function for the *ordered* sequences of blocks, some of them being underlined and the others being non-underlined, i.e., for the *ordered bipartitions.* Furthermore, $(1-G)^{-1} = 1/(3-2e^u)$.

For the compatible bipartitional partitions there are again two kinds of blocks, but this time the underlined blocks must lie to the left of the non-underlined ones. The exponential generating functions for the underlined blocks and for the non-underlined blocks are the same:

$$H = \frac{u^1}{1!} + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots = (e^u - 1).$$

Hence the expansion of $(1-H)^{-1}(1-H)^{-1}$ will be the generating function for the ordered sequences of blocks, the leftmost ones being underlined, the rightmost ones being non-underlined, so that

$$\sum_{r\ge 0} b_r \frac{u^r}{r!} = \Bigl(\frac{1}{(1-(e^u-1))}\Bigr)^2 = \frac{1}{(2-e^u)^2}. \quad \square$$

The sequences (b'_r) and (b_r) do not appear (yet?) in the Sloane integral sequence basis [Sl94]. However our young colleague Jiang Zeng drew our attention to the paper by Knuth [Kn92] who himself pointed out that the generating function $(2-e^u)^{-1}$ already appeared in Cayley (*Collected Math. Papers*, vol. 4, p. 112-115) for enumerating a special class of trees. According to Knuth the coefficients of the Taylor expansion of $(2-e^u)^{-1}$ count the *preferential arrangements of* n *objects.*

2. Notations and first analytic results

We make use of the usual notations: $(a;q)_n$ and $(a;q)_\infty$ for the q-ascending factorials:

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n\ge 1;\end{cases}$$
$$(a;q)_\infty = \lim_n (a;q)_n = \prod_{n\ge 0}(1-aq^n).$$

In particular, $(q)_n = (q;q)_n$ and $(q)_\infty = (q;q)_\infty$.

Recall the q-binomial theorem (see [An76 p. 15] or [GaRa90, § 1.3]) that states

$$\sum_{n\ge 0} \frac{(a;q)_n}{(q;q)_n} u^n = \frac{(au;q)_\infty}{(u;q)_\infty}, \tag{2.1}$$

together with the two q-exponential identities

$$e_q(u)=\sum_{n\ge 0}\frac{u^n}{(q;q)_n}=\frac{1}{(u;q)_\infty};\tag{2.2}$$

$$E_q(u)=\sum_{n\ge 0}\frac{q^{\binom{n}{2}}u^n}{(q;q)_n}=(-u;q)_\infty.\tag{2.3}$$

The q-binomial theorem provides the five expansions (see [An76, p. 15])

$$\sum_{n\ge 0}\begin{bmatrix}s+n\\ n\end{bmatrix}u^n=\frac{1}{(u;q)_{s+1}};\tag{2.4}$$

$$\sum_{n\ge 0}\begin{bmatrix}s\\ n\end{bmatrix}q^{\binom{n+1}{2}}u^n=(-qu;q)_s.\tag{2.5}$$

$$\begin{bmatrix}s+n\\ n\end{bmatrix}=\sum_{s\ge a_1\ge\cdots\ge a_n\ge 0}q^{a_1+\cdots+a_n}=\sum_{n\ge a_1\ge\cdots\ge a_s\ge 0}q^{a_1+\cdots+a_s};\tag{2.6}$$

$$q^{\binom{n}{2}}\begin{bmatrix}s+1\\ n\end{bmatrix}=\sum_{s\ge a_1>\cdots>a_n\ge 0}q^{a_1+\cdots+a_n}.\tag{2.7}$$

$$\frac{1}{(u;q)_{s+1}}=\sum_{n\ge 0}u^n\sum_{n\ge a_1\ge\cdots\ge a_s\ge 0}q^{a_1+\cdots+a_s};\tag{2.8}$$

where the a_i's are non-negative integers.

The ordered bipartition $((B_1,B_2,\ldots,B_k),(\beta_1,\beta_2,\ldots,\beta_k))$ will be kept fixed throughout this section. Let U be the bipartitional relation on $X\times X$ associated with it. Next consider a sequence $\mathbf{c}=(c(1),c(2),\ldots,c(r))$ of non-negative integers; as before, let $v=1^{c(1)}2^{c(2)}\ldots r^{c(r)}$ and denote by $R(v)$ (or by $R(\mathbf{c})$) the class of all rearrangements of the word v. If the block B_l consists of the integers $i_1,i_2,\ldots,i_h$ written in increasing order (with respect to the usual linear order of $X=[\,r\,]$) and if $u_1,u_2,\ldots,u_r$ are r commuting variables, it will be convenient to write

(2.9) $c(B_l)$ for the *sequence* $c(i_1),c(i_2),\ldots,c(i_h)$;

$m_l=\sum c(B_l)$ for the *sum* $c(i_1)+c(i_2)+\cdots+c(i_h)$;

$|\mathbf{c}|$ for the *sum* $c(1)+\cdots+c(r)$ also equal to $m_1+\cdots+m_k$;

$\mathbf{c}\pm 1_i$ for $(c(1),\ldots,c(i-1),c(i)\pm 1,c(i+1),\ldots,c(r))$;

$u(B_l)^{c(B_l)}$ for the *monomial* $u_{i_1}^{c(i_1)}u_{i_2}^{c(i_2)}\ldots u_{i_h}^{c(i_h)}$;

$\sum u(B_l)$ for the *sum* $u_{i_1}+u_{i_2}+\cdots+u_{i_h}$.

$\mathbf{u}^{\mathbf{c}}$ for the *monomial* $u_1^{c_1}u_2^{c_2}\ldots u_r^{c_r}$.

In particular $\binom{m_l}{c(B_l)}$ will denote the multinomial coefficient

$$\binom{c(i_1)+c(i_2)+\cdots+c(i_h)}{c(i_1),c(i_2),\ldots,c(i_h)}.$$

Let $A'_U(q;\mathbf{c}) := \sum_w q^{\mathrm{inv}'_U w}$ $(w \in R(\mathbf{c}))$ denote the generating function for the class $R(\mathbf{c})$ by the statistic "inv'_U."

Proposition 2.1. *With the above notations* (2.9) *the following formulas hold*

$$A'_U(q;\mathbf{c}) = \begin{bmatrix} |\mathbf{c}| \\ m_1,\ldots,m_k \end{bmatrix} \prod_{l=1}^{k} \binom{m_l}{c(B_l)} q^{\beta_l \binom{m_l}{2}}; \tag{2.10}$$

$$\begin{aligned} \sum_{\mathbf{c}} \frac{A'_U(q;\mathbf{c})}{(q)_{|\mathbf{c}|}} \mathbf{u}^{\mathbf{c}} &= \prod_{l;\beta_l=0} e_q\big(\sum u(B_l)\big) \times \prod_{l;\beta_l=1} E_q\big(\sum u(B_l)\big) \\ &= \frac{\prod_{l;\beta_l=1} \big(-\sum u(B_l);q\big)_\infty}{\prod_{l;\beta_l=0} \big(\sum u(B_l);q\big)_\infty}. \end{aligned} \tag{2.11}$$

In the second formula $\mathbf{c}$ *runs over all sequences* $(c(1),\ldots,c(r))$ *with* $c(1) \ge 0, \ldots, c(r) \ge 0$.

Proof. Formula (2.10) follows from the well-known generating function in the ordinary "inv" case. The q-multinomial coefficient is the generating function for the class of words having exactly m_1 letters equal to $1, \ldots, m_k$ letters equal to k by "inv." Such a word gives rise to exactly $\prod_l \binom{m_l}{c(B_l)}$ words in $R(\mathbf{c})$. Finally, the letters belonging to each non-underlined block provide no further U-inversions, while the letters in an underlined block B_l $(\beta_l = 1)$ bring $\binom{m_l}{2}$ extra U-inversions when they are compared between themselves.

To derive (2.11) we have to make use of the traditional q-calculus. First rewrite (2.10) as

$$\frac{A'_U(q;\mathbf{c})}{(q)_{|\mathbf{c}|}} = \prod_{l=1}^{k} \frac{q^{\beta_l\binom{m_l}{2}}}{(q)_{m_l}} \binom{m_l}{c(B_l)}. \tag{2.12}$$

Then the left-hand side of (2.10) is equal to

$$\begin{aligned} &\sum_{c(B_1)} \cdots \sum_{c(B_k)} \frac{A'_U(q;\mathbf{c})}{(q)_{|\mathbf{c}|}} u(B_1)^{c(B_1)} \ldots u(B_k)^{c(B_k)} \\ &\qquad = \prod_{l=1}^{k} \sum_{c(B_l)} \frac{q^{\beta_l\binom{m_l}{2}}}{(q)_{m_l}} \binom{m_l}{c(B_l)} u(B_l)^{c(B_l)} \\ &\qquad = \prod_{l=1}^{k} \sum_{d(l)\ge 0} \frac{q^{\beta_l\binom{d(l)}{2}}}{(q)_{d(l)}} \sum_{m_l=d(l)} \binom{m_l}{c(B_l)} u(B_l)^{c(B_l)} \\ &\qquad = \prod_{l=1}^{k} \sum_{d(l)\ge 0} \frac{q^{\beta_l\binom{d(l)}{2}}}{(q)_{d(l)}} \Big(\sum u(B_l)\Big)^{d(l)}, \end{aligned}$$

which is the right-hand side of (2.11) by using (2.2) and (2.3). ∎

3. An 'Essentially Verification' Manipulative Proof of the 'If' Part of Theorem 1

The generating function according to "maj'_U" does not seem to be directly derivable from the classical, MacMahon formula. Later, we will show that the combinatorial proofs easily carry over, but here we will show a manipulative proof. We will prove the stronger result that the subsets of words with a prescribed last letter have the Mahonian property.

Keep the same notations as in Proposition 2.1. In particular, let $|\mathbf{c}| = c(1)+\cdots+c(r) = m_1+\cdots+m_l$ be the length of the words in the class $R(\mathbf{c})$. Also define, for each letter $i \in X = [\,r\,]$.

$$A'_U(q;\mathbf{c};i) := \sum_w q^{\mathrm{inv}'_U\, w} \qquad (w \in R(\mathbf{c}),\ w \text{ ends with } i)\,;$$

$$A''_U(q;\mathbf{c};i) := \sum_w q^{\mathrm{maj}'_U\, w} \qquad (w \in R(\mathbf{c}),\ w \text{ ends with } i).$$

It is easy to derive a formula for $A'_U(q;\mathbf{c};i)$, in terms of $A'_U(q;\mathbf{c})$. Let i belong to the block B_l $(1 \le l \le k)$. Then,

$$A'_U(q;\mathbf{c};i) = A'_U(q;\mathbf{c}-1_i)q^{m_1+\cdots+m_{l-1}}, \tag{3.1}$$

if B_l is *not* underlined, and

$$A'_U(q;\mathbf{c};i) = A'_U(q;\mathbf{c}-1_i)q^{m_1+\cdots+m_l-1}, \tag{3.2}$$

if B_l is underlined.

Denote by $A''_U(q;\mathbf{c};i)$ the corresponding quantities for "maj'_U." By considering what letter can be second-to-last, we get the following recurrence:

$$A''_U(q;\mathbf{c};i) = q^{|\mathbf{c}|-1} \sum_{j \in B_1\cup\cdots\cup B_{l-1}} A''_U(q;\mathbf{c}-1_i;j) + \sum_{j \in B_l\cup\cdots\cup B_k} A''_U(q;\mathbf{c}-1_i;j) \tag{3.3}$$

when B_l is *not* underlined, and the recurrence

$$A''_U(q;\mathbf{c};i) = q^{|\mathbf{c}|-1} \sum_{j \in B_1\cup\cdots\cup B_l} A''_U(q;\mathbf{c}-1_i;j) + \sum_{j \in B_{l+1}\cup\cdots\cup B_k} A''_U(q;\mathbf{c}-1_i;j) \tag{3.4}$$

when B_l is underlined.

It is a completely routine matter, that we leave to the readers (or rather to their computers) to verify that the expressions on the right sides of (3.1) and (3.2) (using (2.9)) also satisfy the same recurrence. It follows by induction that for all $\mathbf{c}$ and for all $1 \le i \le r$, we have

$$A'_U(q;\mathbf{c};i) = A''_U(q;\mathbf{c};i). \tag{3.5}$$

By summing over i, we get that indeed for bipartitional graphs U the statistics "inv_U" and "maj_U" are equidistributed. $\square$

4. The MacMahon Verfahren

In this section we make use again of the same notations as in section 2. Consider the polynomial

$$A'_U(t,q;\mathbf{c}) = \sum_w t^{\mathrm{des}'_U w} q^{\mathrm{maj}'_U w} \qquad (w \in R(\mathbf{c}))$$

(where $\mathrm{des}'_U w$ is the number of U-descents of the first kind in w defined in the introduction). We first derive the formula
(4.1)

$$\frac{A'_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} = \prod_{l=1}^{k} \binom{m_l}{c(B_l)} \times \sum_{s\geq 0} t^s \prod_{l;\beta_l=0} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \times \prod_{l;\beta_l=1} q^{\binom{m_l}{2}} \begin{bmatrix} s+1 \\ m_l \end{bmatrix}.$$

The previous formula is the *finite version* of (2.11). By multiplying (4.1) by $(1-t)$ and making $t=1$ we recover (2.11). ¿From (4.1) we also derive the *factorial* generating function for the polynomials $A'_U(t,q;\mathbf{c})$ that reads

$$\sum_{\mathbf{c}} \frac{A_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} \mathbf{u}^{\mathbf{c}} = \sum_{s\geq 0} t^s \, \frac{\prod\limits_{l;\beta_l=1} (-\sum u(B_l);q)_{s+1}}{\prod\limits_{l;\beta_l=0} (\sum u(B_l);q)_{s+1}}, \tag{4.2}$$

which is the *finite version* of (2.11).

As done in Proposition 2.1 we can obtain (4.2) from (4.1) by a routine calculation as follows:

$$\begin{aligned}
\sum_{\mathbf{c}} \frac{A'_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} &= \sum_{c(B_1),\dots,c(B_k)} \frac{A'_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} \prod_{l=1}^{k} u(B_l)^{c(B_l)} \\
&= \sum_{c(B_1),\dots,c(B_k)} \prod_{l=1}^{k} u(B_l)^{c(B_l)} \binom{m_l}{c(B_l)} \\
&\qquad \times \sum_{s\geq 0} t^s \prod_{l;\beta_l=0} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \times \prod_{l;\beta_l=1} q^{\binom{m_l}{2}} \begin{bmatrix} s+1 \\ m_l \end{bmatrix} \\
&= \sum_{s\geq 0} t^s \prod_{l;\beta_l=0} \sum_{c(B_l)} u(B_l)^{c(B_l)} \binom{m_l}{c(B_l)} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \\
&\qquad \times \prod_{l;\beta_l=1} \sum_{c(B_l)} u(B_l)^{c(B_l)} \binom{m_l}{c(B_l)} \begin{bmatrix} s+1 \\ m_l \end{bmatrix}
\end{aligned}$$

$$= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \sum_{m_l} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \sum_{\Sigma c(B_l)=m_l} u(B_l)^{c(B_l)} \binom{m_l}{c(B_l)}$$
$$\times \prod_{l;\beta_l=1} \sum_{m_l} \begin{bmatrix} s+1 \\ m_l \end{bmatrix} \sum_{\Sigma c(B_l)=m_l} u(B_l)^{c(B_l)} \binom{m_l}{c(B_l)}$$
$$= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \sum_{m_l} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \Bigl(\sum u(B_l)\Bigr)^{m_l}$$
$$\times \prod_{l;\beta_l=1} \sum_{m_l} \begin{bmatrix} s+1 \\ m_l \end{bmatrix} \Bigl(\sum u(B_l)\Bigr)^{m_l}$$
$$= \sum_{s\ge 0} t^s \frac{\prod_{l;\beta_l=1} (-\sum u(B_l);q)_{s+1}}{\prod_{l;\beta_l=0} (\sum u(B_l);q)_{s+1}},$$

by using the identities (2.4) and (2.5). □

Now we can prove (4.1) using the so-called “MacMahon Verfahren.” As already noted in [Fo95, ClFo95a], the method introduced by MacMahon [Mac13] to derive the generating function for “maj” is to be updated to include a second statistic, but the principle remains the same.

Let $((B_1,\ldots,B_k)(\beta_1,\ldots,\beta_k))$ be the ordered bipartition corresponding to the bipartitional relation U and let $w = x_1x_2\ldots x_m$ be a word of the class $R(\mathbf{c})$, so that $m = |\mathbf{c}|$. Denote by $w(B_l)$ be the *subword* of w consisting of all the letters belonging to B_l $(l = 1,\ldots,k)$. Then replace each letter belonging to B_l by $b_l = \min B_l$ (with respect to the usual order). Call $\overline{w} = \overline{x}_1\overline{x}_2\ldots\overline{x}_m$ the resulting word. Clear the mapping

$$w \mapsto \bigl(\overline{w}, w(B_1), \ldots, w(B_l)\bigr) \tag{4.3}$$

is bijective. Moreover, $\mathrm{des}'_U w = \mathrm{des}'_U \overline{w}$ and $\mathrm{maj}'_U w = \mathrm{maj}'_U \overline{w}$. Accordingly, the polynomial $A'_U(t,q;\mathbf{c})$ is divisible by $\prod_l \binom{m_l}{c(B_l)}$.

For each $i = 1,2,\ldots,m$ let z_i denote the number of U-descents (of the first kind) in the right factor $\overline{x}_i\overline{x}_{i+1}\ldots\overline{x}_m$ of $\overline{w}$. Clearly, $z_1 = \mathrm{des}'_U \overline{w}$ and $z_1+\cdots+z_m = \mathrm{maj}'_U \overline{w}$.

Now let $\mathbf{p} = (p_1,\ldots,p_m)$ be a sequence of m integers satisfying $s' \ge p_1 \ge p_2 \ge \cdots \ge p_m \ge 0$, where s' is a *given* integer. Form the *non-increasing* word $v = y_1y_2\ldots y_m$ defined by $y_i = p_i + z_i$ $(1 \le i \le m)$ and consider the biword

$$\binom{v}{\overline{w}} = \binom{y_1y_2\cdots y_m}{\overline{x}_1\overline{x}_2\ldots\overline{x}_m}.$$

Next rearrange the columns of the previous matrix in such a way that the mutual orders of the columns with the same bottom entries are preserved and the entire bottom row is of the form $b_1^{m_1}b_2^{m_2}\dots b_k^{m_k}$. We obtain the matrix

$$\begin{pmatrix} a_{1,1}\dots a_{1,m_1} & \dots & a_{k,1}\dots a_{k,m_k} \\ b_1\ \dots\ b_1 & \dots & b_k\ \dots\ b_k \end{pmatrix}.$$

By construction each of the k words $a_{1,1}\dots a_{1,m_1}, \dots, a_{k,1}\dots a_{k,m_k}$ is *non-increasing*. Furthermore, if $\overline{x}_i = \overline{x}_{i'}$ and $\overline{x}_i \in B_l$ with l underlined, there is necessarily a U-descent within $\overline{x}_i\overline{x}_{i+1}\dots\overline{x}_{i'}$. Hence $z_i > z_{i'}$ and and $y_i > y_{i'}$. The corresponding word $a_{l,1}\dots a_{l,m_l}$ will then be *strictly decreasing*. Also note that

$$a_{l,i} \le y_1 = p_1 + z_1 \le s' + \mathrm{des}'_U\,\overline{w}$$

for all l, i. Let then $s = s' + \mathrm{des}'_U\,\overline{w}$. It follows that each of the words $a_{l,1}\dots a_{l,m_l}$ satisfies

$$\begin{aligned} &s \ge a_{l,1} \ge \dots \ge a_{l,m_l} \ge 0, \text{ if } l \text{ is } \textit{not} \text{ underlined;} \\ &s \ge a_{l,1} > \dots > a_{l,m_l} \ge 0, \text{ if } l \text{ is underlined.} \end{aligned} \tag{4.4}$$

The mapping $(s', \mathbf{p}, \overline{w}) \mapsto (s, (a_{l,i}))$ is a bijection satisfying

$$\begin{gathered} s = s' + \mathrm{des}'_U\,\overline{w}\,; \\ \sum_{l,i} a_{l,i} = p_1 + \dots + p_m + z_1 + \dots + z_m = \sum_i p_i + \mathrm{maj}'_U\,\overline{w}. \end{gathered} \tag{4.5}$$

Now rewrite (2.8) as

$$\frac{1}{(t;q)_{m+1}} = \sum_{s\ge 0} \sum_{s\ge p_1\ge\dots\ge p_m\ge 0} q^{p_1+\dots+p_m},$$

so that by (4.3) we have

$$\begin{aligned} \frac{1}{\prod_l \binom{m_l}{c(B_l)}} \frac{A'_U(t,q;\mathbf{c})}{(t;q)_{m+1}} &= \sum_{s\ge 0} t^s \sum_{s\ge p_1\ge\dots\ge p_m\ge 0} q^{\Sigma\, p_i} \sum_{\overline{w}} t^{\mathrm{des}'_U\,\overline{w}} q^{\mathrm{maj}'_U\,\overline{w}} \\ &= \sum_{s',\mathbf{p},\overline{w}} t^{s'+\mathrm{des}'_U\,\overline{w}} q^{\Sigma\, p_i + \mathrm{maj}'_U\,\overline{w}} = \sum_{(s,(a_{l,i}))} t^s q^{\Sigma a_{l,i}} \qquad \text{[by (4.5)]} \\ &= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \sum_{s\ge a_{l,1}\ge\dots\ge a_{l,m_l}\ge 0} q^{a_{l,1}+\dots+a_{l,m_l}} \\ &\qquad\qquad \times \prod_{l;\beta_l=1} \sum_{s\ge a_{l,1}>\dots>a_{l,m_l}\ge 0} q^{a_{l,1}+\dots+a_{l,m_l}} \\ &= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \prod_{l;\beta_l=1} q^{\binom{m_l}{2}} \begin{bmatrix} s+1 \\ m_l \end{bmatrix} \qquad \text{[by (2.6) and (2.7).]} \end{aligned}$$

Hence (4.1) is established. $\square$

As (4.2) implies (2.10) and as the latter identity holds in the U-number-of-inversion version, we have another proof of the 'if' part of Theorem 1.

5. The bijective proof of Theorem 1

Let U be a bipartitional relation. In this section we construct a bijection Φ_U of each class $R(\mathbf{c})$ onto itself satisfying

$$\text{(5.1)} \qquad \operatorname{maj}'_U w = \operatorname{inv}'_U \Phi_U(w).$$

One of the main ingredients in the construction of Φ_U is the second fundamental transformation Φ (see, e.g., [Lo83, chap. 10]) that satisfies

$$\text{(5.2)} \qquad \operatorname{maj} w = \operatorname{inv} \Phi(w)$$

on each rearrangement class. The bijection Φ_U is the *conjugate* of Φ in the sense that we have

$$\text{(5.3)} \qquad \Phi_U = \delta^{-1} \circ \Phi \circ \delta,$$

for a certain bijection δ.

Let us first recall the construction of Φ [Lo83, chap. 10]: let w be a word in the alphabet X and $x \in X$. Two cases are to be considered

(i) the last letter of w is greater than x;

(ii) the last letter of w is at most equal to x.

In case (i) let $(w_1x_1, w_2x_2, \ldots, w_hx_h)$ be the factorization of w having the following properties: $x_1, x_2, \ldots, x_h$ are letters of X *greater* than x and $w_1, w_2, \ldots, w_h$ are words all letters of which are less than or equal to x.

In case (ii) $x_1, x_2, \ldots, x_h$ are letters of X at most equal to x, while $w_1, w_2, \ldots, w_h$ are words all letters of which are greater than x.

Call *x-factorization* the above factorization. In both cases we have

$$w = w_1x_1w_2x_2 \ldots w_hx_h\,;$$

then define

$$\gamma_x\, w = x_1w_1x_2w_2 \ldots x_hw_h.$$

The construction of Φ goes as follows. If w is of length 0 or 1, let $\Phi(w) = w$. For a word wx with $x \in X$ and w of positive length, form $\Phi(w)$ (already defined by induction), apply γ_x to $\Phi(w)$ and add x at the end of the resulting words, i.e., define

$$\Phi(wx) = \big(\gamma_x\, \Phi(w)\big)x.$$

Property (5.2) was proved in [Fo68] (see also [Lo83, chap. 10]). We shall make use of two further properties.

Proposition 5.1.

(i) *Both w and $\Phi(w)$ end with the same letter.*

(ii) *Let y and y' be two adjacent letters (with respect to the usual order) in the alphabet X and suppose that both occur exactly once in w. Then, if y occurs to the left of y' in w, the same holds for $\Phi(w)$.*

Property (i) is true by the very definition of Φ. Property (ii) requires a simple verification that will be left out. $\square$

Let $((B_1,\dots,B_k)(\beta_1,\dots,\beta_k))$ be the ordered bipartition corresponding to the bipartitional relation U. We keep the notations given in § 2. If w is a word in $R(\mathbf{c})$, let $m_l=\sum B_l$ be the number of letters in w belonging to B_l and let $w(B_l)$ be the *subword* of w consisting of all the letters belonging to B_l $(l=1,\dots,k)$.

The conjugation δ is defined as follows.

(i) For every $l=1,\dots,k$ replace each letter belonging to B_l by $b_l=\min B_l$ (with respect to the usual order). Call $\overline{w}$ the resulting word.

(ii) If l is *non-underlined*, read $\overline{w}$ *from left to right* and replace the successive occurrences of b_l by $(b_l,1)$, $(b_l,2)$, … , (b_l,m_l); do this for each non-underlined l.

(iii) Do the operation described in (ii) for each *underlined* l, but this time read $\overline{w}$ *from right to left*.

The word derived after all those operations will be denoted by w_U. It is actually a rearrangement of the word $(b_1,1)\dots(b_1,m_1)\dots(b_k,1)\dots(b_k,m_k)$ (all letters distinct.) Furthermore, w_U contains the subword $(b_l,1)\dots(b_l,m_l)$ (resp. $(b_l,m_l)\dots(b_l,1)$) if l is non-underlined (resp. underlined). To be able to define "maj" for w_U we need a *linear* order on those ordered pairs. We shall take:

$$(b_l,j)>(b_{l'},j') \text{ iff } l<l' \text{ or } l=l' \text{ and } j>j'. \tag{5.3}$$

The conjugation δ is then defined by

$$\delta: w\mapsto (w_U, w(B_1), w(B_2),\dots,w(B_r)). \tag{5.4}$$

The inverse map δ^{-1} simply consists of replacing each *subword*

$$(b_l,1)(b_l,2)\dots(b_l,m_l)\qquad (\text{resp. } (b_l,m_l)\dots(b_l,2)(b_l,1))$$

within w_U by the subword $w(B_l)$.

Lemma 5.2. *With "maj" defined by means of the total order* (5.3) *the following identity holds:*

$$\operatorname{maj}'_U w=\operatorname{maj} w_U. \tag{5.5}$$

Proof. Let $w = x_1x_2\dots x_m$ and $w_U = z_1z_2\dots z_m$ (the letters z_i are ordered pairs (b_l, j)). If $(x_i, x_{i+1}) \in U$, then either $x_i \in B_l$, $x_{i+1} \in B_{l'}$ with $l < l'$, or x_i, x_{i+1} are both in the same *underlined* block B_l.

In the first case, $z_i = (b_l, j)$ and $z_{i+1} = (b_{l'}, j')$ for some j, j'. But as $l < l'$, we have $z_i > z_{i+1}$ by (5.3). In the second case, $z_i = (b_l, j)$ and $z_{i+1} = (b_l, j')$; but as l is underlined we have $j > j' = j - 1$ and again $z_i > z_{i+1}$.

Now if $(x, x_{i+1}) \notin U$, then either $x_i \in B_l$, $x_{i+1} \in B_{l'}$ with $l > l'$, or x_i, x_{i+1} are both in the same *non-underlined* block B_l. In the first case the same argument as above shows that $z_i < z_{i+1}$. In the second case the labelling from left to right of the non-underlined letters of $\overline{w}$ yields $z_i = (b_l, j) < (b_l, j+1) = z_{i+1}$. ∎

Next apply the second fundamental transformation to w_U. We obtain a rearrangement $\Phi(w_U)$ that statisfies

$$\operatorname{maj} w_U = \operatorname{inv} \Phi(w_U). \tag{5.6}$$

Lemma 5.3. *For each $l = 1, 2, \dots, k$ both words w_U and $\Phi(w_U)$ contain the subword*

$$(b_l, 1)(b_l, 2)\dots(b_l, m_l) \qquad (\text{resp. } (b_l, m_l)\dots(b_l, 2)(b_l, 1))$$

depending on whether l is non-underlined or underlined.

Proof. This is a consequence of Proposition 5.1 (ii). ∎

Finally, if we apply the conjugation δ^{-1} to $\Phi(w_U)$ using the subwords $w(B_1), \dots, w(B_k)$, we obtain a rearrangement $\delta^{-1}\,\Phi(w_U)$ which is a rearrangement of the original word w and satisfies

$$\operatorname{inv} \Phi(w_U) = \operatorname{inv}_U \delta^{-1}\,\Phi(w_U). \tag{5.7}$$

We shall denote it by $\Phi_U(w)$. All the above transformations are reversible. The product $\Phi_U = \delta^{-1} \circ \Phi \circ \delta$ is a well-defined bijection of $R(\mathbf{c})$ onto itself satisfying (5.1).

6. A proof of the 'only if' part of Theorem 1

The proof of that "only if" part will be the consequence of the following sequence of lemmas.

Lemma 6.1. *If there exists an element $x \in X$ such that $U \subset (X \setminus \{x\}) \times (X \setminus \{x\})$ and $U \neq \emptyset$, then the equidistribution of* inv_U *and* maj_U *does not hold.*

Proof. Let w be a word having no letter equal to x and let v be a word in the class $R(x^m w)$ $(m \geq 1)$. Denote by $\overline{v}$ the word derived from a word v by deleting all its letters equal to x. Then $\operatorname{inv}_U v = \operatorname{inv}_U \overline{v}$. On the other hand, $\operatorname{maj}_U x^m\overline{v} = m \times \operatorname{des}_U \overline{v} + \operatorname{maj}_U \overline{v}$. As U is non-empty, there exists a rearrangement class $R(w)$ and a word $w' \in R(w)$ such that $\operatorname{des}_U w' \geq 1$. Thus there is a bound b such that for every $m \geq 1$ and for every $v \in R(x^m w)$ we have $\operatorname{inv}_U v \leq b$, while

$$\max_{v \in R(x^m w)} \operatorname{maj}_U v \geq \operatorname{maj}_U x^m \overline{v} \geq m. \quad \square$$

Lemma 6.2. *If $(x,y) \in U$, $(y,x) \in U$ and $x \neq y$, and if the equidistribution of* inv_U *and* maj_U *holds, then*

$$(x,x) \in U \quad \text{and} \quad (y,y) \in U.$$

Proof. Suppose $(x,x) \notin U$. In the class $R(x^2y)$ we have $\operatorname{inv}_U w = 2$ for all w, while $\operatorname{maj}_U xyx = 3$ and the equidistribution does not hold for $R(x^2y)$. $\square$

Let $X = \{x,y,z\}$; at this stage it would be useful to have a thorough table of the relations U on $X \times X$ for which the equidistribution of inv_U and maj_U holds. As there are six elements in $X \times X \setminus \operatorname{diag} X \times X$, there would be only sixty-four cases to consider. As there are many symmetries, the table could be rapidly set up. A verification by computer could also be used. We have preferred to verify the property in each case.

Lemma 6.3. *If (x,y), (y,x), (x,z) and (z,x) belong to U, if x, y and z are different and if the equidistribution of* inv_U *and* maj_U *holds, then U contains the product $\{x,y,z\} \times \{x,y,z\}$.*

Proof. In other words, besides (x,x), (y,y) and (z,z) (as shown in Lemma 6.2), the relation U must also contain (y,z) and (z,y).

If the conclusion does not hold, there are three cases to be studied. The distributions of inv_U and maj_U on the rearrangement class $R(xyz)$ are shown in the next table and are never identical. $\square$

	$(y,z) \notin U, (z,y) \in U$		$(y,z) \in U, (z,y) \notin U$		$(y,z) \notin U, (z,y) \notin U$	
w	inv_U	maj_U	inv_U	maj_U	inv_U	maj_U
xyz	2	1	3	3	2	1
xzy	3	3	2	1	2	1
yxz	2	3	3	3	2	3
yzx	2	2	3	3	2	3
zxy	3	3	2	3	2	3
zyx	3	3	2	2	2	2

If U is a relation on $X \times X$, its *symmetric* part, i.e., the set of all ordered pairs (x, y) such that both (x, y) and (y, x) belong to U, is denoted by $S(U)$. Also let $A(U) = U \setminus S(U)$ be its *asymmetric part.* Finally, let X_U be the subset of X of all the x's such that $(x, y) \in S(U)$ (and so $(y, x) \in S(U)$) for some $y \in X$.

Lemma 6.4. *If the equidistribution holds for U, then $S(U)$ is an equivalence relation on $X_U \times X_U$.*

Proof. Let $x \in X_U$ and let $y \in X$ such that $(x, y) \in S(U)$. If $y = x$, then $(x, x) \in U$. If $y \neq x$, Lemma 6.2 also implies that $(x, x) \in U$. Thus $S(U)$ is reflexive. By definition, $S(U)$ is symmetric. Now let $x, y, z \in X_U$ and suppose $(x, y) \in S(U)$ and $(y, z) \in S(U)$. Then Lemma 6.3 implies that $(x, y) \in S(U)$. The relation is then transitive. $\square$

Thus, if the equidistribution holds for U, there is a partition $\{B_1, \ldots, B_l\}$ of X_U such that $S(U) = B_1 \times B_1 \cup \cdots \cup B_l \times B_l$. The subsets $B_1, \ldots, B_l$ will be called the *blocks of* X_U.

Lemma 6.5. *Suppose that the equidistribution holds for U and let x, y be two distinct elements belonging to the same block, say, B_i of X_U and let z be an element of X. Then*

$$(z, x) \in U \Leftrightarrow (z, y) \in U\,;$$
$$(x, z) \in U \Leftrightarrow (y, z) \in U.$$

Proof. If $z \in B_i$, then $(x, z) \in S(U)$ and $(y, z) \in S(U)$ and there is nothing to prove. If z belongs to another block B_j of X_U and if $(z, x) \in U$, then $(x, z) \notin U$. Otherwise, we would have $B_i = B_j$. If $z \notin X_u$ and $(z, x) \in U$, again $(x, z) \notin U$. Otherwise, $(x, z) \in S(U)$ and this would contradict $z \notin X_U$.

Suppose that conditions $(z, x) \in U$ and $(z, y) \notin U$ hold. Two cases are to be considered : (a) $(y, z) \notin U$; (b) $(y, z) \in U$.

In case (a) we have $(x, y), (y, x), (z, x) \in U$, $(x, z), (z, y), (y, z) \notin U$. But for each word w in the class $R(xyz)$ we have $\operatorname{inv}_U w \leq 2$, while $\operatorname{maj}_U zxy = 3$, so that the equidistribution would not hold.

In case (b) we have $(x, y), (y, x), (z, x), (y, z) \in U$, $(x, z), (z, y) \notin U$. Let $V = X \times X \setminus U$. Then $\operatorname{inv}_V w \leq 2$ for all w, while $\operatorname{maj}_V xzy = 3$. Thus, the equidistribution would not hold for V and also for U.

Thus cases (a) and (b) cannot occur and consequently if $(z, y) \in U$ holds, we must have $(z, y) \in U$. The elements x and y play a symmetric role, so that the first equivalence is proved.

The proof of the second equivalence is quite analogous. If $(x, z) \in U$, we have seen that $(z, x) \notin U$. Suppose $(y, z) \notin U$ and consider the two cases : (a) $(z, y) \notin U$; (b) $(z, y) \in U$.

In case (a) we have $(x, y), (y, x), (x, z) \in U$, $(z, x), (y, z), (z, y) \notin U$. Again for each $w \in R(xyz)$ we have $\mathrm{inv}_U\, w \leq 2$, while $\mathrm{maj}_U\, yxz = 3$, so that the equidistribution does not hold.

In case (b) we have $(x, y), (y, x), (x, z), (z, y) \in U$, $(z, x), (y, z) \notin U$. Let $V = X \times X \setminus U$. Then $\mathrm{inv}_V\, w \leq 2$, while $\mathrm{maj}_V\, yzx = 3$, so that the equidistribution does not hold for V, and then for U.

As x and y play a symmetric role, the second equivalence is also established. $\square$

Notation. It will be convenient to write $x \to y$ for $(x, y) \in U$, $x \not\to y$ or $y \not\leftarrow x$ for $(x, y) \notin U$, $x \rightleftharpoons y$ for $(x, y) \in S(U)$.

Lemma 6.6. *Suppose that the equidistribution holds for U. Then, either there is a block B_1 of X_U with the property*
$\forall\, x \in B_1, \forall\, y \in X \setminus B_1$, *then* $(x, y) \notin U$;
or there exists an $x \in X \setminus X_U$ such that
$\forall\, y \in X$, *then* $(x, y) \notin U$.

The foregoing property means that by rearranging the elements of X, either the top left corner $B_1 \times (X \setminus B_1)$ of $X \times X$, or a left block $C \times X$ has no intersection with U.

Proof. Suppose that the conclusion is false. This means that for every block B_i of X_U there is $x \in B_i$, $y \in X \setminus B_i$ such that $(x, y) \in U$ and also that for all $x \in X \setminus X_U$ there is $y \in X \setminus \{x\}$ such that $(x, y) \in U$.

Let $x_0 \in X$. If x_0 belongs to a block B_{i_0}, there is $x_1 \in B_{i_0}$ and $x_2 \notin B_{i_0}$ such that $x_0 \rightleftharpoons x_1 \to x_2$. But the previous lemma says that :

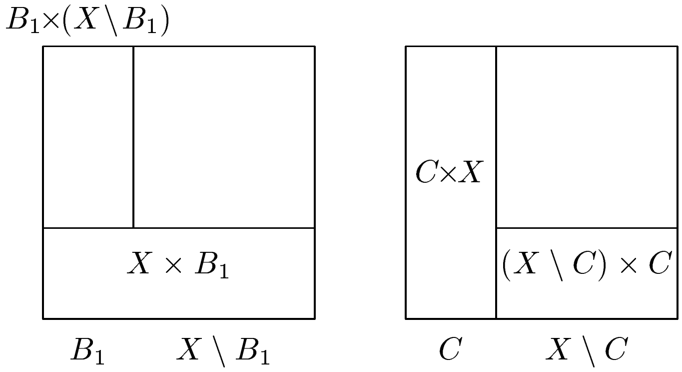

Fig. 2

if $x_0 \rightleftharpoons x_1$ and $x_1 \to x_2$, then $x_0 \to x_2$. Also $x_2 \not\to x_0$.
Now, either $x_2 \in B_{i_2}$ with $i_2 \neq i_1$ or $x_2 \notin X_U$. In the first case there is $x_3 \in B_{i_2}$ and $x_4 \notin B_{i_2}$ such that $x_2 \rightleftharpoons x_3 \to x_4$. Using the same lemma we also have $x_2 \to x_4$ and $x_4 \not\to x_2$. If $x_2 \notin X_U$, there is $x_4 \neq x_2$ such that $x_2 \to x_4$ and also $x_4 \not\to x_2$, because $x_2 \notin X_U$.

We can then build a sequence $(x_0, x_2, x_4, \dots)$ with the property

$$x_0 \to x_2 \to x_4 \to x_6 \to \cdots$$
$$x_0 \not\leftarrow x_2 \not\leftarrow x_4 \not\leftarrow x_6 \not\leftarrow \cdots$$

and such that $x_{2i} \neq x_{2i+2}$ at each step. If we had started with an element $x_0 \notin X_U$, the conclusion would have been the same.

The above sequence cannot be infinite and have all its elements distinct, so that, after relabelling, there is a finite sequence $(y_1, y_2, \dots, y_{n+1})$ of elements of X with the following properties :

(a) $n \geq 2$;
(b) all terms $y_1, y_2, \dots, y_n$ are different ;
(c) $y_1 \to y_2 \to y_3 \to \cdots \to y_n \to y_{n+1} = y_1$;
(d) $y_1 \not\leftarrow y_2 \not\leftarrow y_3 \not\leftarrow \cdots \not\leftarrow y_n \not\leftarrow y_{n+1} = y_1$;

If $n = 2$, we have $y_1 \to y_2 \to y_1$ and $y_1 \not\leftarrow y_2 \not\leftarrow y_1$, a contradiction, so that $n \geq 3$.

Consider the class $R(y_1 y_2 \dots y_n)$. Then $\mathrm{maj}_U\, y_1 y_2 \dots y_n = 1 + 2 + \cdots + (n-1) = n(n-1)/2$. If there is a word $v = z_1 z_2 \dots z_n \in R(y_1 y_2 \dots y_n)$ such that $\mathrm{inv}_U\, v = n(n-1)/2$, this means that

$$\begin{aligned} z_1 \to z_2,\ z_1 \to z_3,\ \dots, z_1 \to z_n \\ z_2 \to z_3,\ \dots, z_2 \to z_n \\ \cdots \\ z_{n-1} \to z_n \end{aligned} \tag{$\star$}$$

If $z_1 = y_i$ with $2 \leq i \leq n+1$, let j be the unique integer such that $z_j = y_{i-1}$. Relation (e) above says that $z_j = y_{i-1} \not\leftarrow y_i = z_1$. This contradicts ($\star$). $\square$

Let $C = \{x \in X \setminus X_U : \forall\, y \in X,\ x \not\to y\}$. From Lemma 6.6 it follows that, if the equidistribution hods and C is empty, there is a unique block B_1 of X_U such that

$$(\star\star) \qquad \forall\, x \in B_1,\ \forall\, y \in X \setminus B_1, \quad \text{then } x \not\to y.$$

Lemma 6.7. *Suppose that the equidistribution holds for U. If C is non-empty, then*

$$(\star\star\star) \qquad (X \setminus C) \times C \subset U.$$

In other words, $\forall\, y \in X \setminus C$, $\forall\, x \in C$, then $y \to x$.

If C is empty and if B_1 is the block defined in $(\star\star)$, then

$$(iv) \qquad (X \setminus B) \times B \subset U.$$

In other words, $\forall\, y \in X \setminus B$, $\forall\, x \in B$, then $y \to x$.

See Fig. 1 : in each case the bottom rectangle to the right is entirely contained in U.

Proof. Assume that C is non-empty and suppose that $(\star\star\star)$ does not hold. Then there is $y \in X \setminus C$ and also $x \in C$ such that $y \not\to x$ and $y \neq x$. As $y \notin C$, there is $z \in X$ such that $y \to z$. Notice that z may be equal to y, if $y \in X_U$, but $z \neq x$, as we have assumed $y \not\to x$.

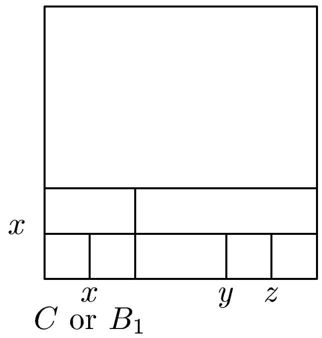

Fig. 3

Consider the class $R(xyz)$. By assumption, $y \not\to x$, $y \to z$ and also $x \not\to y$, $x \not\to z$, since $x \in C$. Four cases are to be considered :

(a) $z \to x$, $z \to y$; (b) $z \to x$, $z \not\to y$; (c) $z \not\to x$, $z \to y$; (d) $z \not\to x$, $z \not\to y$.

In both cases (a) and (b) $\mathrm{inv}_U\, w \leq 2$ for all w, while $\mathrm{maj}_U\, yzx = 3$. In both cases (c) and (d) $\mathrm{inv}_U\, w \leq 1$ for all w, while $\mathrm{maj}_U\, xyz = 2$. Thus there is never equidistribution on $R(xyz)$.

Suppose that C is empty. Let B_1 be the block defined in $(\star\star)$. If (iv) does not hold, there is $y \in X \setminus B_1$ and also $x \in B_1$ such that $y \not\to z$. As $y \notin B_1$ and since C is supposed to be empty, there exists z such that $y \to z$. Again we have $y \not\to x$, $y \to z$, $x \not\to y$, $x \not\to z$. The same analysis as above shows that there is no equidistribution on $R(xyz)$. ∎

It follows from Lemma 6.6 and Lemma 6.7 that, if the equidistribution holds for U, then, either C is non-empty and then

$C \times X$ is empty and $(X \setminus C) \times C \subset U$,

or C is empty and then there is a unique block B_1 of X_U such that

$B_1 \times (X \setminus B_1)$ is empty and $(X \setminus B_1) \times \subset U$.

The theorem is now easily proved by induction on card X. If the equidistribution holds for U defined on $X \times X$ and if C is non-empty, then the equidistribution also holds for the relation $V = U \cap (X \setminus C) \times (X \setminus C)$ defined on $(X \setminus C) \times (X \setminus C)$. By induction V is bipartitional. Hence, U is also bipartitional (see Fig. 1).

In the same manner, if C is empty, then the equidistribution holds for the relation $V = U \cap (X \setminus B_1) \times (X \setminus B_1)$. By induction V is bipartitional. Hence, U is also bipartitional.

7. Compatible bipartitional relations

The statistics "maj_U" and "inv_U" have been defined in (0.2); also remember that "des_U" counts the U-descents of the second kind, as defined at the end of the introduction. The calculation of the generating function for $(\mathrm{des}_U, \mathrm{maj}_U)$ and the construction of the bijection that carries "maj_U" onto "inv_U" will be very similar to their equivalent derivations for $(\mathrm{des}'_U, \mathrm{maj}'_U)$, "$\mathrm{maj}'_U$" and "$\mathrm{inv}'_U$." Let

$$A_U(q;\mathbf{c}) = \sum_w q^{\mathrm{inv}_U w} \qquad (w \in R(\mathbf{c}))\,; \tag{7.1}$$

$$A_U(t,q;\mathbf{c}) = \sum_w t^{\mathrm{des}_U w} q^{\mathrm{maj}_U w} \qquad (w \in R(\mathbf{c})). \tag{7.2}$$

The identity

$$A_U(q;\mathbf{c}) = \begin{bmatrix} |\mathbf{c}| \\ m_1,\ldots,m_k \end{bmatrix} \prod_{l=1}^{k} \binom{m_l}{c(B_l)} q^{\beta_l \binom{m_l+1}{2}}; \tag{7.3}$$

follows from (2.10), as we have to add the total number of underlined letters, i.e., $\sum\limits_l \beta_l m_l$ to the power of q.

The proof of the formula

$$\begin{aligned} \sum_{\mathbf{c}} \frac{A_U(q;\mathbf{c})}{(q)_{|\mathbf{c}|}} \mathbf{u}^{\mathbf{c}} &= \prod_{l;\beta_l=0} e_q\big(\textstyle\sum u(B_l)\big) \times \prod_{l;\beta_l=1} E_q\big(q \textstyle\sum u(B_l)\big) \\ &= \frac{\prod\limits_{l;\beta_l=1} (-q\sum u(B_l); q)_\infty}{\prod\limits_{l;\beta_l=0} (\sum u(B_l); q)_\infty}\,; \end{aligned} \tag{7.4}$$

follows the same pattern as the proof of (2.11).

Let $A_U^{\mathrm{maj}}(q;\mathbf{c}) = \sum\limits_w q^{\mathrm{maj}_U w}$ $(w \in R(\mathbf{c}))$. Again, we don't prove that $A_U^{\mathrm{maj}}(q;\mathbf{c})$ is equal to the right-hand side of (7.3). We'd rather derive the formulas for $A_U(t,q;\mathbf{c})$, defined in (7.2), in the spirit of section 4.

Proposition 7.1. *Let U be a compatible bipartitional relation. Then*

$$\begin{aligned} &\frac{A_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} \\ &\quad = \prod_{l=1}^{k} \binom{m_l}{c(B_l)} \times \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \times \prod_{l;\beta_l=1} q^{\binom{m_l+1}{2}} \begin{bmatrix} s \\ m_l \end{bmatrix}; \end{aligned} \tag{7.5}$$

$$\sum_{\mathbf{c}} \frac{A_U(t,q;\mathbf{c})}{(t;q)_{|\mathbf{c}|+1}} \mathbf{u}^{\mathbf{c}} = \sum_{s\ge 0} t^s \frac{\prod\limits_{l;\beta_l=1} (-q\sum u(B_l); q)_s}{\prod\limits_{l;\beta_l=0} (\sum u(B_l); q)_{s+1}}. \tag{7.6}$$

Proof. Let $w \mapsto \big(\overline{w}, w(B_1), \ldots, w(B_l)\big)$ and $(s', \mathbf{p}, \overline{w}) \mapsto (s, (a_{l,i}))$ be the two bijections defined in section 4. We keep the same notations as in that section. In particular, let $\overline{w} = \overline{x}_1\overline{x}_2\ldots\overline{x}_m$. The only difference to be brought to the constructions of those bijections is to notice that $z_m = 1$ iff $\overline{x}_m$ belongs to an underlined block. Consequently, the sequences $a_{l,1}\ldots a_{l,m_l}$ associated with the *underlined* blocks are still *strictly* decreasing, but also $a_{l,m_l} \ge 1$.

The reason is the following: let l be underlined and let $\overline{x}_i$ be the *rightmost* letter of $\overline{w}$ that belongs to the block B_l. If $i = m$, then $a_{l,m_l} = p_m + z_m \ge 1$; if $i < m$, then, either there is one *non-underlined* letter in the factor $\overline{x}_{i+1}\ldots\overline{x}_m$ and necessarily one U-descent because U is supposed to be *compatible*, or all the letters in that factors are underlined and in particular $z_m = 1$. In both cases, $a_{l,m_l} \ge 1$.

Accordingly, the mapping $(s', \mathbf{p}, \overline{w}) \mapsto (s, (a_{l,i}))$ is a bijection satisfying

$$s \ge a_{l,1} \ge \cdots \ge a_{l,m_l} \ge 0, \text{ if } l \text{ is } \textit{not} \text{ underlined;}$$
$$s \ge a_{l,1} > \cdots > a_{l,m_l} \ge 1, \text{ if } l \text{ is underlined.;}$$
$$s = s' + \operatorname{des}_U \overline{w}\,;$$
$$\sum_{l,i} a_{l,i} = \sum_i p_i + \operatorname{maj}_U \overline{w}.$$

In the same manner as in section 4 we have

$$\begin{aligned}
\frac{1}{\prod_l \binom{m_l}{c(B_l)}} \frac{A_U(t,q;\mathbf{c})}{(t;q)_{m+1}} &= \sum_{s\ge 0} t^s \sum_{s\ge p_1\ge\cdots\ge p_m\ge 0} q^{\Sigma p_i} \sum_{\overline{w}} t^{\operatorname{des}_U \overline{w}} q^{\operatorname{maj}_U \overline{w}} \\
&= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \sum_{s\ge a_{l,1}\ge\cdots\ge a_{l,m_l}\ge 0} q^{a_{l,1}+\cdots+a_{l,m_l}} \\
&\qquad \times \prod_{l;\beta_l=1} \sum_{s\ge a_{l,1}>\cdots>a_{l,m_l}\ge 1} q^{a_{l,1}+\cdots+a_{l,m_l}} \\
&= \sum_{s\ge 0} t^s \prod_{l;\beta_l=0} \begin{bmatrix} m_l+s \\ m_l \end{bmatrix} \prod_{l;\beta_l=1} q^{\binom{m_l+1}{2}} \begin{bmatrix} s \\ m_l \end{bmatrix},
\end{aligned}$$

by (2.6) and (2.7). $\square$

As (7.4) holds (in the U-number-of-inversion version) and since (7.6) implies (7.4), we then have a proof of the 'if' part of Theorem 2.

The proof of the 'only if' part is straightforward. Suppose that U is non-compatible, so that there is an underlined block B_l to the left of a non-underlined one $B_{l'}$, i.e., $l < l'$. Take two integers $x \in B_l$ and $x'_l \in B_{l'}$ and consider the class $R(xx')$ of the two words xx' and $x'x$. Then $\operatorname{inv}_U xx' = 2$, $\operatorname{inv}_U x'x = 1$, while $\operatorname{maj}_U xx' = 3$, $\operatorname{maj}_U x'x = 0$.

8. A bijective Proof of Theorem 2

Let $\pi = (\underline{B_1}, \ldots, \underline{B_n}, B_{n+1}, \ldots, B_r)$ be a compatible ordered bipartition having exactly n underlined blocks lying in the beginning and let U be the corresponding compatible bipartitional relation. As done in the papers by Steingrìmsson [St93] and Clarke and Foata (*op. cit.*), let us introduce an extra letter $\star$ and form the new compatible ordered bipartition

$$\pi^\star = (\underline{B_1}, \ldots, \underline{B_n}, \{\star\}, \{B_{n+1}\}, \ldots, \{B_r\}). \tag{8.1}$$

Denote by $U^\star$ the bipartitional relation associated with $\pi^\star$. Notice that $U^\star$ is a relation on $(X \cup \{\star\}) \times (X \cup \{\star\})$. We now make use of the transformation $\Phi_{U^\star}$ (constructed in section 5) on the words in the alphabet $X \cup \{\star\}$.

If the word $w = x_1 x_2 \ldots x_m$ belongs to the class $R(\mathbf{c})$, form the word $w\star$. Its image under $\Phi_{U^\star}$ will yield a word of the form $w'\star$, by Proposition 5.1. There is a $U^\star$-descent at position m, if and only if x_m is underlined. Hence

$$\mathrm{maj}'_{U^\star}\, w\star = \mathrm{maj}'_U\, w + m\,\chi(x_m \text{ is underlined}). \tag{8.2}$$

Also adding $\star$ at the end of w' will increase the number of $U^\star$-inversions by exactly the number of underlined letters in w', i.e., $|w'|_-$. Hence

$$\mathrm{inv}'_{U^\star}\, w'\star = \mathrm{inv}'_U\, w' + |w'|_-. \tag{8.3}$$

Hence
$$\begin{aligned}
\mathrm{maj}_U\, w &= \mathrm{maj}'_U\, w + m\,\chi(x_m \text{ is underlined}) && [\text{by } (0.2)]\\
&= \mathrm{maj}'_{U^\star}\, w\star && [\text{by } (8.2)]\\
&= \mathrm{inv}'_{U^\star}\, w'\star && [\text{by } (5.1)]\\
&= \mathrm{inv}'_U\, w' + |w'|_- && [\text{by } (0.2)]\\
&= \mathrm{inv}_U\, w'. && [\text{by } (8.3)]
\end{aligned}$$

As $\Phi_{U^\star}$ maps the set of all words in each rearrangement class ending with $\star$ onto the same set, the mapping $w \mapsto w'$ is a bijection of $R(\mathbf{c})$ onto itself. Moreover, it satisfies

$$\mathrm{maj}_U\, w = \mathrm{inv}_U\, w'. \tag{8.4}$$

Remark: Formulas (4.2) and (7.6) are the factorial generating functions for the pairs $(\mathrm{des}'_U, \mathrm{maj}'_U)$ and $(\mathrm{des}_U, \mathrm{maj}_U)$, respectively. On the other hand, the bijection Φ_U (defined in § 5) and the bijection $w \mapsto w'$ just defined (that we shall denote by Ψ_U) satisfy (4.1) and (8.4). Let $?'_U = \mathrm{des}'_U \circ \Phi_U^{-1}$ and $?_U = \mathrm{des}_U \circ \Psi_U^{-1}$, so that

$$\begin{aligned}
(\mathrm{des}'_U, \mathrm{maj}'_U)\, w &= (?'_U, \mathrm{inv}'_U)\, \Phi_U(w)\,;\\
(\mathrm{des}_U, \mathrm{maj}_U)\, w &= (?_U, \mathrm{inv}_U)\, \Psi_U(w).
\end{aligned}$$

Th natural question arises: can we find suitable predicates to define "$?'_U$" and "$?_U$" without any references to the bijections Φ_U and Ψ_U?

References

[An76] George E. Andrews, "*The Theory of Partitions*." London, Addison-Wesley, 1976 (*Encyclopedia of Math. and Its Appl.*, **2**).

[Be94] François Bergeron, Gilbert Labelle, Pierre Leroux, "*Théorie des espèces et combinatoire des structures arborescentes*." Montréal, Université du Québec à Montréal, Publ. du LACIM, no. 19, 1994.

[ClFo94] Robert J. Clarke and Dominique Foata, "Eulerian Calculus, I: univariable statistics," *Europ. J. Combinatorics*, **15**, 1994, p. 345–362.

[ClFo95a] Robert J. Clarke and Dominique Foata, Eulerian Calculus, II: an extension of Han's fundamental transformation, to appear in *Europ. J. Combinatorics*, 1995.

[ClFo95b] Robert J. Clarke and Dominique Foata, Eulerian Calculus, III: the ubiquitous Cauchy formula, to appear in *Europ. J. Combinatorics*, 1995.

[Co70] Louis Comtet, "*Analyse Combinatoire*, vol. 1 and 2." Paris, Presses Universitaires de Frances, 1970. (English edition *Advanced Combinatorics*, D. Reidel, Dordrecht, 1974).

[Fo68] Dominique Foata, "On the Netto inversion number of a sequence," *Proc. Amer. Math. Soc.*, **19**, 1968, p. 236–240.

[Fo74] Dominique Foata, "*La série génératrice exponentielle dans les problèmes d'énumération*." Montréal, Presses Universitaires de Montréal, 1974.

[Fo95] Dominique Foata, Les distributions Euler-Mahoniennes sur les mots, to appear in *Discrete Math.*, 1995.

[GaRa90] George Gasper and Mizan Rahman, "*Basic Hypergeometric Series*." London, Cambridge Univ. Press, 1990 (*Encyclopedia of Math. and Its Appl.*, **35**).

[Han95] Guo-Niu Han, Ordres bipartitionnaires et statistiques sur les mots, to appear in *Electronic J. Combinatorics*, 1995.

[Kn73] Donald E. Knuth, "*The Art of Computer Programming*, vol. 3, *Sorting and Searching*." Addison-Wesley, Reading, 1973.

[Kn92] Donald E. Knuth, "Convolutional Polynomials," *Mathematica J.*, **2**, 1992, p. 67–78.

[Lo83] M. Lothaire, "*Combinatorics on words*." Reading, Addison-Wesley, 1983 (*Encyclopedia of Math. and its Appl.*, **17**).

[Mac13] P.A. MacMahon, "The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects," *Amer. J. Math.*, **35**, 1913, p. 314–321.

[Mac15] P.A. MacMahon, "*Combinatory Analysis*, vol. 1." Cambridge, Cambridge Univ. Press, 1915 (Reprinted by Chelsea, New York, 1955).

[Mac78] P.A. MacMahon, *Collected Papers*, vol. 1 [G.E. Andrews, ed.]. Cambridge, Mass., The M.I.T. Press, 1978.

[Sl94] N.J.A. Sloane, "*A Handbook of Integer Sequences.*" New York, Academic Press, 1973.

[St93] Einar Steingrìmsson, "Permutation Statistics of Indexed Permutations," *Europ. J. Combinatorics*, **15**, 1994, p. xxx–xxx.