The ABSTRACT LACE EXPANSION

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For Erdős Pál, In Memoriam

Abstract: David Brydges and Thomas Spencer's Lace Expansion is abstracted, and it is shown how it sometimes gives rise to sieves.

LACES

Definition: Let P be a finite set of *properties*. A mapping l that assigns to any subset $S \subset P$ another subset $l(S)$, is called a *lace-map*, if for all $S, G, S_1, S_2 \subset P$:
(i) $l(S) \subset S$; (ii) $l(S) \subset G \subset S \Rightarrow l(G) = l(S)$; (iii) $l(S_1) = l(S_2) \Rightarrow l(S_1 \cup S_2) = l(S_1)$.

A set L for which $l(L) = L$ is called a *lace*. By applying (ii) to $G = l(S)$, it is seen that $l(l(S)) = l(S)$, for any set of properties S, hence $l(S)$ is always a lace, and l is a projection: $l^2 = l$.

If L is a lace then, by (iii), there exists a set $C(L) \subset P \backslash L$ such that

$$\{S \subset P \mid l(S) = L\} = \{S \mid L \subset S \subset L \cup C(L)\} \quad .$$

The set $C(L)$ is called the *set of properties compatible with* L. The collection of laces will be denoted by $\mathcal{L}$. For any lace L, obviously $C(L) = \{p \in P \backslash L \mid l(L \cup \{p\}) = L\}$.

Theorem: Let X be a set of elements each of which possesses a subset of the properties of P. Let wt be any function on X (in particular the counting function $wt(x) \equiv 1$). For any lace L define $N(L)$ to be the sum of the weights of those elements of X that *definitely have* all the properties of L and *definitely don't have* any of the properties in $C(L)$. Then the sum of the weights of those elements of X that have none of the properties of P, $N_0(X)$, is given by:

$$N_0(X) = \sum_{L \in \mathcal{L}} (-1)^{|L|} N(L) \quad . \qquad (Lace\_Expansion)$$

Proof: For each property $p \in P$, assign a variable Y_p. Since every subset S of P has a unique lace $L = l(S)$, and by (iii), the collection of subsets G for which $l(G) = L$ consist of the interval (in the Boolean lattice) between L and $L \cup C(L)$, we have

$$\prod_{p \in P} (1 + Y_p) = \sum_{S \subset P} \prod_{s \in S} Y_s = \sum_{L \in \mathcal{L}} \sum_{\substack{S; \\ l(S)=L}} \prod_{s \in S} Y_s =$$

$$\sum_{L \in L} \prod_{s \in L} Y_s \sum_{\substack{S; \\ l(S)=L}} \prod_{s \in S \backslash L} Y_s = \sum_{L \in \mathcal{L}} \prod_{s \in L} Y_s \prod_{s \in C(L)} (1 + Y_s) \quad . \qquad (*)$$

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For every $x \in X$ and $p \in P$, let $\chi_p(x) = -1$ if x has property p and 0 if x doesn't. Since $0^r = 0$ when $r > 0$, while $0^0 = 1$, we have

$$N_0(X) = \sum_{nX} wt(x) \prod_{p \in P} (1 + \chi_p(x)) = \sum_{x \in X} wt(x) \sum_{L \in \mathcal{L}} \prod_{p \in L} \chi_p(x) \prod_{p \in C(L)} (1 + \chi_p(x)) =$$

$$\sum_{L \in \mathcal{L}} \sum_{x \in X} wt(x) \prod_{p \in L} \chi_p(x) \prod_{p \in C(L)} (1 + \chi_p(x)) = \sum_{L \in \mathcal{L}} (-1)^{|L|} N(L) \quad \square \, .$$

SIEVES

Let's call a lace *saturated* if its set of compatible properties, $C(L)$, is empty. In this case $N(L)$ is simply the sum of the weights of the elements of X that definitely have all the properties in L (and possibly others). Let $\mathcal{L}_s$ be the collection of saturated laces. If l is such that the parity of the cardinalities of all unsaturated laces is always the same, one has the inequalities:

$$N_0(X) \leq \sum_{L \in \mathcal{L}_s} (-1)^{|L|} N(L) \quad \textit{or}$$

$$N_0(X) \geq \sum_{L \in \mathcal{L}_s} (-1)^{|L|} N(L) \ ,$$

according to whether that parity is odd or even respectively.

EXAMPLES

1) The identity lace-map : $l(S) = S$ for every $S \subset P$. Every subset S of P is a lace, and $C(S)$ is always empty. In this case the Lace Expansion reduces to the inclusion-exclusion principle.

2) The Bonferroni Lace: Let $P = \{1, \ldots, n\}$, for a positive integer n. For $S \subset P$, define $l(S)$ to be the subset of S consisting of its k smallest elements, if $|S| \geq k$, and otherwise $l(S) := S$. It is easy to see that l is a lace-map, and that the laces are the subsets of P with cardinality $\leq k$. The saturated laces are those whose cardinality is $< k$ and for an unsaturated lace $L = \{i_1 < \ldots < i_k\}$, $C(L) = i_k + 1, i_k + 2, \ldots, n$. The resulting sieve is the Bonferroni sieve (see, e.g., [C]).

3) The Brun Lace : Let $P = \{1, 2, \ldots, n\}$, and let $n \geq N_1 \geq N_2 \geq \ldots \geq N_n \geq 1$ be given beforehand. For a set $S = \{i_1 > i_2 > \ldots > i_r\}$, define $l(S) = S$ if $i_1 \leq N_1, i_2 \leq N_2, \ldots, i_r \leq N_r$, and otherwise, let $l(S) = \{i_1, \ldots, i_s\}$, where s is the smallest index such that $i_s > N_s$. It is easy to see that l is a lace-map. The saturated laces are sets of the form $L = \{i_1 > \ldots > i_r\}$, where $i_1 \leq N_1, i_2 \leq N_2, \ldots, i_r \leq N_r$. The unsaturated laces are sets $L = \{i_1 > \ldots > i_r\}$, such that $i_1 \leq N_1, i_2 \leq N_2, \ldots, i_{r-1} \leq N_{r-1}$, but $i_r > N_r$. For such a lace, $C(L) = \{i_r - 1, i_r - 2, \ldots, 1\}$.

To get upper and lower *Brun sieves*([B]) we must have $N_1 = N_2, N_3 = N_4, \ldots N_{n-1} = N_n$, and $N_2 = N_3, N_4 = N_5, \ldots N_{n-1} = N_n$, respectively.

4) The Brydges-Spencer Original Lace Expansion[BS] (see [MS] for a very lucid exposition): Fix n, and let $P = \{(i, j); 0 \leq i < j \leq n\}$. It is instructive to think of $n + 1$ dots placed in a row, at locations $0, 1, \ldots, n$. Then P is the set of all arcs joining two dots. For any set of arcs G, the lace-map, $l(G)$, is defined as follows. Let $s_1 := 0$, and let t_1 be the largest t such that $0t \in G$. Among all the arcs $st \in G$ that "go over" t_1, let t_2 be the endpoint that goes farthest amongst them: $t_2 := max\{t : st \in G, s < t_1\}$, and amongst those arcs take s_2 to be the left-endpoint farthest to the left that connects to t_2: $s_2 := min\{s : st_2 \in G\}$. One continues recursively: $t_i := max\{t : st \in G, s < t_{i-1}\}$, and $s_i := min\{s : st_i \in G\}$, until either one gets, for some k, $t_k = n$, or one finishes the *connected component* and starts a new component with the next dot. The lace of G, $l(G)$, is defined to be the collection of these arcs $\{s_1 t_1, s_2 t_2 \ldots, s_k t_k\}$. Observe that the arcs of an irreducible lace interlace, forming the kind of lace people embroider, which explains its name.

The Brydges-Spencer lace expansion does not yield any sieves, but serves another purpose: to determine the asymptotic behavior of the number of n-step self-avoiding walks. Brydges and Spencer[BS] determined the asymptotics for *weakly avoiding* walks in dimensions $d \geq 5$, while Hara and Slade[HS][MS], in one of the greatest mathematical feats of this decade, strengthened it to the regular self-avoiding walk in $d \geq 5$. The problem is still open for dimensions $d = 2, 3, 4$. Perhaps we need a more complicated lace-map.

SPECULATIONS

I predict that the Abstract Lace Expansion (ALE) has a bright future. The very powerful *Probabilistic Method*([ASE],[S]) uses the Bonferroni sieve with $k = 2$. Introducing appropriate lace-maps may make it even more powerful. The *Satisfiability Problem* (the grandmother of all NP-complete problems) can be approached via counting (the number of covered 0-1 vectors), and introducing powerful lace-maps may improve the average-running-time of current algorithms. Another possible application is to improving current asymptotic upper and lower bounds for $R(n,n)$, as well as to the exact evaluations of $R(5,5)$ and $R(6,6)$, in spite of Paul Erdős's pessimistic prophesy (see [S], p. 4). Recall that the Ramsey number, $R(n,n)$, is the smallest N such that if you 2-color the edges of the complete graph on N vertices, then you are *guaranteed* a monochromatic K_n.

This gives rise to the following counting problem. Let X is the set of all $2^{\binom{N}{2}}$ edge-colorings, and let P consist of the $\binom{N}{n}$ properties: $A_S :=$ the induced coloring on S is monochromatic, where S ranges over all $n-$subsets of the set of vertices $\{1, \ldots, N\}$. Find the number $N_0(X)$ of property-less colorings. If, thanks to some lower sieve, we can ascertain that $N_0(X) > 0$, then we would get that $R(n,n) > N$. If, on the other hand, thanks to an upper sieve, we would find that $N_0(X) \leq 0$, then we would know that $N_0(X) = 0$, and that $R(n,n) \leq N$. So we need good lace-maps that would give good sieves that, in turn, would make the lower and upper bounds zero-in at the exact value.

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