There Are MORE THAN 2(n/17) n-LETTER TERNARY SQUARE-FREE WORDS**

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Abstract: We prove that the 'connective constant' for ternary square-free words is at least $2^{1/17} = 1.0416\ldots$, improving on Brinkhuis and Brandenburg's lower bounds of $2^{1/24} = 1.0293\ldots$ and $2^{1/21} = 1.033\ldots$ respectively. This is the first improvement since 1983.

A word is *square-free* if it never stutters, i.e. if it cannot be written as axxb for words a,b and non-empty word x. For example, 'example' is square-free, but 'exampample' is not. See Steven Finch's famous Mathematical Constants site[4] for a thorough discussion and many references. Let $a(n)$ be the number of ternary square-free n-letter words (**A006156, M2550** in the Sloane-Plouffe[5] listing, $1, 3, 6, 12, 18, 30, 42, \ldots$). Brinkhuis[3] and Brandenburg[2] (see also [1]) showed that $a(n) \geq 2^{n/24}$, and $a(n) \geq 2^{n/21}$ respectively. Here we show, by extending the method of [3], that $a(n) \geq 2^{n/17}$, and hence that $\mu := \lim_{n\to\infty} a(n)^{1/n} \geq 2^{1/17} = 1.0416\ldots$.

Definition: A triple-pair $[\,[\,U_0, V_0\,]\,,\,[\,U_1, V_1\,]\,,\,[\,U_2, V_2\,]\,]$ where $U_0, V_0, U_1, V_1, U_2, V_2$ are words in the alphabet $\{0, 1, 2\}$ of the same length k, will be called a *k-Brinkhuis triple-pair* if the following conditions are satisfied.

• The 24 words of length $2k$,

$$[U \text{ or } V]_0[U \text{ or } V]_1\,,[U \text{ or } V]_0[U \text{ or } V]_2\,,[U \text{ or } V]_1[U \text{ or } V]_2\,,$$

$$[U \text{ or } V]_1[U \text{ or } V]_0\,,[U \text{ or } V]_2[U \text{ or } V]_0\,,[U \text{ or } V]_2[U \text{ or } V]_1\,,$$

(i.e. U_0U_1, $U_0V_1, \ldots, V_2V_1$), are all square-free.

• For every length r, $k/2 \leq r < k$, the 12 words consisting of the heads and tails of $\{U_0, U_1, U_2, V_0, V_1, V_2\}$ of length r are all distinct. □

It is easy to see (do it from scratch, or adapt the argument in [2]), that if $[\,[\,U_0, V_0\,]\,,\,[\,U_1, V_1\,]\,,\,[\,U_2, V_2\,]\,]$ is a k-Brinkhuis triple-pair, then for every square-free word $x = x_1 \ldots x_n$ of length n in the alphabet $\{0, 1, 2\}$, the 2^n words of length nk, $[U \text{ or } V]_{x_1}[U \text{ or } V]_{x_2} \ldots [U \text{ or } V]_{x_n}$ are also all square-free. Thus the mere existence of a k-Brinkhuis triple-pair implies that $a(nk) \geq 2^n a(n)$, which implies that $\mu \geq 2^{1/(k-1)}$.

Theorem: The following is an 18-Brinkhuis triple-pair

[[2 1 0 2 0 1 2 0 2 1 2 0 1 0 2 0 1 2 , 2 1 0 2 0 1 0 2 1 2 0 2 1 0 2 0 1 2],

[0 2 1 0 1 2 0 1 0 2 0 1 2 1 0 1 2 0 , 0 2 1 0 1 2 1 0 2 0 1 0 2 1 0 1 2 0],

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[1 0 2 1 2 0 1 2 1 0 1 2 0 2 1 2 0 1 , 1 0 2 1 2 0 2 1 0 1 2 1 0 2 1 2 0 1]].

Proof: Purely Routine! □

Remark: The above 18-Brinkhuis triple-pair was found by the first author by running procedure `FindPair();` in the Maple package `JAN`, written by the second author. `JAN` is available from the second author's website `http://www.math.temple.edu/~zeilberg/` (Click on `Maple programs and packages`, and then on `JAN`.)

Another Remark: Brinkhuis[3] constructed a 25-Brinkhuis triple-pair in which U_0 and V_0 were palindromes, and U_1, U_2, were obtained from U_0 by adding, component-wise, 1 and 2 mod 3, respectively, and similarly for V_1, V_2. Our improved example resulted from relaxing the superfluous condition of palindromity, but we still have the second property. It is very likely that by relaxing the second property, it would be possible to find even shorter Brinkhuis triple-pairs, and hence get yet better lower bounds for μ. Alas, in this case the haystack gets much larger!

References

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