“TRIVIALIZING” GENERALIZATIONS OF SOME IZERGIN-KOREPIN-TYPE DETERMINANTS

TEWODROS AMDEBERHAN, DORON ZEILBERGER

Tulane Univ., Dept of Math., New Orleans, LA 70118, USA
Rutgers Univ., Dept of Math., Piscataway, NJ 08854, USA

ABSTRACT. We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson Condensation and/or Krattenthaler's factor exhaustion method.

All our matrices will be assumed to be embedded inside an infinite matrix.

The first theorem adds *parameters* to the determinant formulas found in Kuperberg [Ku] (Theorem 15), as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laksov-Lascoux-Thorup, and Tsuchiya [T]. This way, the formulation is suited to the method of [AZ]. Our proofs are much more **succinct** and **automatable**, since their generality enables an easy induction using Dodgson's rule [D, AZ], or by employing Krattenthaler's elegant factor exhaustion method [Kr1]. Relevant background for this paper can found in [Ku], and references thereof.

Theorem 1:

$$\det\left(\frac{1}{x_i+y_j+Ax_iy_j}\right)_{i,j}^{1,n} = \frac{\prod_{i<j}(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j+Ax_iy_j)}.$$

$$\det\left(\frac{1}{x_i+y_j}-\frac{1}{1+x_iy_j}\right)_{i,j}^{1,n} = \frac{\prod_{i<j}(1-x_ix_j)(1-y_iy_j)(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j)(1+x_iy_j)}\cdot\prod_{i=1}^{n}(1-x_i)(1-y_i).$$

$$\det\left(\frac{Ay_j+Bx_i}{y_j+x_i}\right)_{i,j}^{1,n} = (A-B)^{n-1}\frac{\left(A\prod_j^n y_j+(-1)^{n-1}B\prod_i^n x_i\right)\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}^{n}(x_i+y_j)}.$$

$$\det\left(\frac{1-x_iy_j}{y_j-x_i}\right)_{i,j}^{1,n}$$
$$=(-1)^{\binom{n}{2}}\frac{\left(\prod_j^n(1+x_j)(1-y_j)+\prod_i^n(1-x_i)(1+y_i)\right)\prod_{i<j}(x_i-x_j)(y_i-y_j)}{2\prod_{i,j}^n(y_j-x_i)}.$$

Sketch of Proof: An automatic application of Dodgson Condensation [D, AZ]. □

Corollary (Cauchy, Stembridge, Laksov-Lascoux-Thorup)

$$\det\left(\frac{1}{x_i+y_j}\right)_{i,j}^{1,n}=\frac{\prod_{i<j}(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^n(x_i+y_j)}.$$

$$Pf^2=\det\left(\frac{x_j-x_i}{x_j+x_i}\right)_{i,j}^{1,2n}=\prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(x_i+x_j)^2}.$$

$$Pf^2=\det\left(\frac{x_j-x_i}{1-x_ix_j}\right)_{i,j}^{1,2n}=\prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(1-x_ix_j)^2}.$$

Remark: Notice that the latter two statements apply only to even-dimensional matrices. An error from [Ku] in the second formula has been corrected here. Pf stands for Pfaffian of a matrix.

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 [Ku] (typos corrected in [Kr2, Theorems 13 and 14]). Below, Z_1, Z_2, Z_3, Z_4, Z_5 are defined by (here $\gamma(a,b)=a-b, \tau(a,b)=a+b$)

$$Z_1(p,q;x,y)_{i,j}=\frac{\gamma(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})},\qquad Z_2(p,q;x,y)_{i,j}=\frac{\gamma(q^{1+j-i},x^{1+j-i})}{\tau(p^{1+j-i},y^{1+j-i})},$$
$$Z_3(p,q;x,y)_{i,j}=\frac{\gamma(q^{-1+j-i},x^{-1+j-i})}{\tau(p^{-1+j-i},y^{-1+j-i})},\qquad Z_4(q,q;x,x)_{i,j}=\frac{\gamma(q^{a+j-i},x^{a+j-i})}{\tau(q^{a+j-i},x^{a+j-i})},$$
$$Z_5(q,q;x,x)_{i,j}=\frac{\tau(q^{b+j-i},x^{b+j-i})}{\gamma(q^{b+j-i},x^{b+j-i})},$$

for $a\in\mathbb{Z}, b=\pm n,\pm(n+1),\ldots$.

Let $\delta_{e,n}=\frac{1+(-1)^n}{2}$ denote *Kronecker's delta* function centered at the even integers, and let $\lambda_{i,j}=1$, if $i\neq j$ and $\lambda_{i,i}=0$.

Theorem 2: *Write $\gamma(a,b)=a-b, \tau(a,b)=a+b$, we have the matrix determinants*

$$\det\left(\frac{\gamma(q^{n+j-i},x^{n+j-i})}{\gamma(p^{n+j-i},y^{n+j-i})}\right)_{i,j}^{1,n}=(py)^{\binom{n}{3}}\gamma(q,x)^n\frac{\prod_{j>i}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{i,j}\gamma(p^{n+j-i},y^{n+j-i})}.$$

$$\det\left(\frac{\tau(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})}\right)_{i,j}^{1,n}=\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\prod_{2\nmid j-i>0}\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{(qx)^{\lfloor n^2/4\rfloor}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_1)_{i,j}^{1,n} = \delta_{e,n}\frac{\gamma(q,x)^n(py)^{n/2}}{(qx)^{\lfloor n^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_2)_{i,j}^{1,n} = \frac{\gamma(q,x)^n}{(qx)^{\lfloor (n-1)^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\tau(p^n,y^n)^{1-\delta_{e,n}}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_3)_{i,j}^{1,n} = \frac{(-py)^n\gamma(q,x)^n}{(qx)^{\lfloor (n+1)^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\tau(p^n,y^n)^{1-\delta_{e,n}}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_4)_{i,j}^{1,n} = 2^{n-1}\frac{q^{na}+(-1)^n x^{na}}{(qx)^{n(n-1)(n+1-3a)/6}}\frac{\prod_{j\neq i}\gamma(q^{|j-i|},x^{|j-i|})}{\prod_{i,j}\tau(q^{a+j-i},x^{a+j-i})}.$$

$$\det(Z_5)_{i,j}^{1,n} = (-1)^{\binom{n}{2}}2^{n-1}\frac{q^{nb}+x^{nb}}{(qx)^{n(n-1)(n+1-3b)/6}}\frac{\prod_{j\neq i}\gamma(q^{|j-i|},x^{|j-i|})}{\prod_{i,j}\gamma(q^{b+j-i},x^{b+j-i})}.$$

$$Pf^2 = \det\left(\lambda_{i,j}\frac{\gamma(q^{j-i},x^{j-i})\gamma(r^{j-i},z^{j-i})}{\gamma(p^{j-i},y^{j-i})}\right)_{i,j}^{1,2n} = \frac{(yp)^n}{(qxrz)^{n^2}}\frac{\prod_{j\neq i}^{1,n}\gamma(p^{|j-i|},y^{|j-i|})^2}{\prod_{i,j}^{1,n}\gamma(p^{n+j-i},y^{n+j-i})^2}\times$$
$$\prod_{i,j}^{1,n}\gamma(qp^{|j-i|},xy^{|j-i|})\gamma(xp^{|j-i|},qy^{|j-i|})\gamma(rp^{|j-i|},zy^{|j-i|})\gamma(zp^{|j-i|},ry^{|j-i|}).$$

Sketch of Proof: Identities Z_4 and Z_5 are directly amenable to Dodgson's Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader. □

Acknowledgment: The authors wish to thank Christian Krattenthaler for helpful suggestions.

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