GERT ALMKVIST'S GENERALIZATION OF A MISTAKE OF BOURBAKI [1]

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Dedicated to the memory of Emil Grosswald

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Theorem([A1]): Let f be an $n \times n$ matrix whose entries are drawn from a commutative ring A. $f^{m+1} = 0$ implies $(Tr(f))^{nm+1} = 0$ [3] [4]

Proof([A1]): Let $e_1, ..., e_n$ be the standard basis of A^n, and think of f as a linear transformation defined on the basis elements by

$$f(e_i) = \sum_{j=1}^{n} f_{i,j} e_j \quad , and\, extended\, by\, linearity \ .$$

Consider the algebra ΛA^n [5] generated by the "indeterminates" $e_1, ..., e_n$ with coefficients in the ring A and product $\wedge$ defined by associativity, distributivity, and the commutation relations $e_i \wedge e_j = -e_j \wedge e_i$. In particular, $e_i \wedge e_i = 0$, for any e_i, and hence any "monomial" $e_{i_1} \wedge ... \wedge e_{i_r}$ that has $i_a = i_b$ for some $1 <= a < b <= r$ (i.e. has repeated factors) must vanish.

It is easily seen that

$$(Tr\, f)(e_1 \wedge ... \wedge e_n) \ = \ \sum_{i=1}^{n} e_1 \wedge ... \wedge f(e_i) \wedge ... \wedge e_n \quad , \tag{$*$}$$

[1] A long time ago, Gert Almkvist told me that one day he will tell me why he is known among his friends and colleagues as "the guy who generalized a mistake of Bourbaki". He finally fulfilled his promise, and I was so excited about his beautiful proof, that I decided to share it with the rest of the mathematical world, especially since his proof is buried inside a very technical paper([A1]). This mistake lead Almkvist to the foundation of the algebraic K-theory of endomorphisms in [A2], that contains the result that there are no other additive invariants of an endomorphism of a finitely generated projective module (over any commutative ring) than the characteristic polynomial (and the class of the module in $K(0, A)$).

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[3] This generalizes, and corrects the following mistake in Bourbaki's [B] p. 156, ex. 13d: "if f is a 2×2 matrix then $f^2 = 0$ implies $(Tr\, f)^4 = 0$ and 4 is the smallest integer with this property". Almkvist's theorem shows that 4 should be replaced by 3.

[4] Note that if A is an algebraically closed field, then the result follows immediately from the Jordan canonical form, since a nilpotent matrix is strictly upper-triangular w.r.t to some basis, and hence trace-less. However Almkvist's result is true for any commutative ring. Note also that if f is diagonal (or even upper-triangular), then Almkvist's theorem follows from the multinomial theorem. Note also that $nm + 1$ is best possible, since if f is the diagonal matrix diag$(x_1, ..., x_n)$, with ${x_i}^{m+1} = 0\, but\, , {x_i}^m \neq 0, (i = 1, ..., n)$, then $(Tr\, f)^{nm} = ((nm)!/m!^n)(x_1...x_n)^m$, which in general is not zero.

[5] The so-called exterior (or Grassmann) algebra

and it follows by anti-symmetry and multi-linearity that for any vectors $b_1, ..., b_n$ in A^n (i.e. the b's are linear combinations of the e's, with coefficients in A):

$$(Tr\, f)(b_1 \wedge ... \wedge b_n) \; = \; \sum_{i=1}^{n} b_1 \wedge ... \wedge f(b_i) \wedge ... \wedge b_n \quad . \qquad (**)$$

By starting with (*), and iterating according to (**), it follows that for any positive integer N, $(Tr\, f)^N (e_1 \wedge ... \wedge e_n)$ is a linear combination of terms of the form $f^{\alpha_1} e_1 \wedge ... \wedge f^{\alpha_n} e_n$, with $\alpha_1 + ... + \alpha_n = N$. [6] But if $N = nm + 1$, each such term is 0, since at least one of the α_i is bound to be $> m$. QED

Almkvist[A1] used the same idea to prove a more general theorem. [7] Other applications of the method include $det(fg) = det(f)det(g)$, [8] Jacobi's $det(exp(f)) = exp(Tr(f))$, [9] the Cayley-

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6 More precisely, we have the "multinomial" theorem

$$(Tr\, f)^N (e_1 \wedge ... \wedge e_n) = \sum_{\alpha_1 + ... + \alpha_n = N} \frac{N!}{\alpha_1! ... \alpha_n!} f^{\alpha_1} e_1 \wedge ... \wedge f^{\alpha_n} e_n \; ,$$

since (**) says that with the correspondence

$$f^{\alpha_1} e_1 \wedge ... \wedge f^{\alpha_n} e_n \rightarrow x_1^{\alpha_1} ... x_n^{\alpha_n} \quad ,$$

multiplying from the left by $Tr\, f$ is like multiplying by $(x_1 + ... + x_n)$.

7 $(Tr\, f)$ can be thought of as the sum of all the 1×1 principal minors of f. Let $a_r(f)$ be the sum of all the $r \times r$ principal minors, then Almkvist's more general theorem says that ${a_1}^{nu_1} \cdots {a_n}^{nu_n} = 0$, whenever $nu_1 + 2nu_2 + ... + nnu_n > mn$. To prove this Almkvist first observed that (*) generalizes to

$$a_r(f)(e_1 \wedge ... \wedge e_n) \; = \sum_{1 <= i_1 < ... < i_r <= n} e_1 \wedge ... \wedge f(e_{i_1}) \wedge ... \wedge f(e_{i_r}) \wedge ... \wedge e_n \quad , \qquad (*')$$

and the obvious generalization of (**), with the e_i in (*') replaced by general b_i also holds by multilinearity, so by iterating, ${a_1}^{nu_1} \cdots {a_n}^{nu_n}$ is an integer-linear combination of terms of the form $f^{\alpha_1} e_1 \wedge ... \wedge f^{\alpha_n} e_n$, with $\alpha_1 + ... + \alpha_n = nm + 1$, so must vanish, just as before.

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$$det(fg)(e_1 \wedge ... \wedge e_n) \; = \; fg(e_1) \wedge ... \wedge fg(e_n) \; = \; det(f) g(e_1) \wedge ... \wedge g(e_n) \; = \; det(f)det(g) e_1 \wedge ... \wedge e_n \quad .$$

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$$exp(Tr(f))(e_1 \wedge ... \wedge e_n) \; = \; \sum_{r=0}^{\infty} (1/r!) Tr(f)^r (e_1 \wedge ... \wedge e_n) = \sum_{r=0}^{\infty} (1/r!) \sum_{\alpha_1 + ... \alpha_n = r} \frac{r!}{\alpha_1! ... \alpha_n!} f^{\alpha_1} e_1 \wedge ... \wedge f^{\alpha_n} e_n \; =$$

Hamilton-type theorem (a_r as in footnote 6) [10]

$$\sum_{r=0}^{m}(-1)^r a_r(f)Tr(f^{m-r}) \; = \; 0 \; , \qquad (C-H)$$

and Rosset's [R] beautiful proof of the Amitsur-Levitski theorem. [11]

$$\sum_{\alpha_1>=0,...,\alpha_n>=0} \frac{1}{\alpha_1!...\alpha_n!} f^{\alpha_1}e_1 \wedge ... \wedge f^{\alpha_n}e_n \; = \; exp(f)e_1 \wedge ... \wedge exp(f)e_n \; = \; det(exp(f))e_1 \wedge ... \wedge e_n \; .$$

Jacobi's identity can be used to give another proof of Almkvist's theorem (but not of its generalization in footnote 6): Since $exp(tf)$ is a polynomial of degree m in t, $det(exp(tf))$ is a polynomial of degree mn, and hence the coefficient of t^{mn+1} in $exp(tTr(f))$ vanishes.

[10] Going back to the correspondence with multinomials of footnote 5, (*') and its "b-analog" imply that multiplying from the left by $a_r(f)$ corresponds to multiplying by the elementary symmetric function

$$E_r(x_1, ..., e_n) \; = \sum_{1<=i_1<...<i_r<=n} x_{i_1}...x_{i_r} \; ,$$

and multiplying by $Tr(f^r)$, corresponds to multiplying by the power functions

$$P_r(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^r \; .$$

It follows that any polynomial identity in $Tr(f^r)$ and $a_r(f)$, holds if it holds when $Tr(f^r)$ is replaced by P_r and $a_r(f)$ is replaced by E_r. But then the identity becomes Newton's celebrated identities (see [Z] for a combinatorial proof). In fact, Newton's identities are the special case when f is upper-triangular. When A is an algebraically closed field, the general identity follows from the Jordan canonical form, but the above argument is simpler, and holds *for any commutative ring A.*

[11] The Amitsur-Levitski identity (see [R]) states that if $f_1, ..., f_{2n}$ are $n \times n$ matrices whose entries belong to a commutative ring A, then

$$\sum_{\pi \epsilon S_{2n}} sgn(\pi) f_{\pi(1)}...f_{\pi(2n)} \; = \; 0.$$

Rosset's [R] ingenious proof goes as follows. Let f be the following matrix, (whose entries are linear combinations of e_i)

$$f := (f_1)e_1 + ... + (f_{2n})e_{2n} \; ,$$

(Do not confuse $(f_1)e_1$ with $f_1(e_1)$. The former is a matrix with elements that are (linear) exterior

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forms, while the latter is an exterior linear form itself.) The Amitsur-Levitski identity is equivalent to the statement that $f^{2n} = 0$. Rosset now claim that $Tr(f^{2r}) = 0$, for $r > 0$. Since $Tr(gh) = Tr(hg)$, it follows that the trace of any product of matrices is not changed when the leftmost matrix is moved to the extreme right. The above operation, when applied to a product of matrices of even length, takes even permutations into odd ones (since we multiplied the underlying permutation by a cycle of even length, which has sign -1). The coefficient of a typical term $e_{i_1} \wedge ... \wedge e_{i_2}r$ in f^{2r} is

$$\sum_{\pi \epsilon S_{2r}} sgn(\pi) f_{i_{\pi(1)}} ... f_{i_{\pi(2r)}} \quad .$$

It is clear that its trace is zero, since the terms can be arranged in mutually canceling pairs, as explained above.

Now, the entries of f do not commute, but the entries of f^2 do! They are linear combinations of pairs $e_i \wedge e_j$, and since $(-1)^2 = +1$, all such pairs mutually commute. Now since $Tr((f^2)^r) = 0$, $r = 1, 2, ..., n$, it follows from (C-H) that $a_r(f^2) = 0$, $r = 1, ..., n$, which implies $(f^2)^n = 0$ by the Cayley-Hamilton theorem.