Doron Gepner's Statistics on Words in $\{1,2,3\}^*$ is (Most Probably) Asymptotically Logistic

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Dedicated to my friend and hero, Doron Gepner (b. March 31, 1956), on his 60^{th} birthday. "Doron le Doron me-Doron".

Preface

I first met Doron Gepner in 1980, when he was a Physics graduate student at the Weizmann Institute of Science, and I was a young *khoker bakhir*. Already then Doron was a legend, since he was the first person in Israel, as far as I know, to have solved *Rubik's cube* completely from scratch, using group-theoretical methods. I was so impressed that I asked him to present a guest-lecture in my graduate combinatorics class, and the students loved it.

Doron then went on to do seminal work in theoretical physics, that, unfortunately, is over my head. But the part that is really interesting to me is his current work, greatly generalizing the celebrated *Rogers-Ramanujan* identities, and giving lots of new insight. I am sure that this work will lead to many future gems.

The purpose of the present note is to present a *present* (*Doron* in Hebrew)[1], from one Doron to another, by paying an old *debt*. In 1987, when he was a postdoc at Princeton University, Doron introduced a new *permutation statistic* (see below), that came up in his work in string theory and conformal field theory. It so happened that at the time, my friend and collaborator, the eminent French combinatorialist, Dominique Foata, visited me. Foata is the world's greatest expert on permutation (and word-) statistics (and coined the term!), so it was only natural that we both got intrigued and tried to investigate Gepner's new statistics, that we christened *gep*, in analogy with the classical statistics *inv*, *maj*, and *des* (see below). We had some preliminary results, but not enough for a paper. This was due to the fact that my beloved servant, Shalosh B. Ekhad, was not yet born. Now, almost thirty years later, it is a good opportunity to revisit Doron Gepner's difficult statistics and harness the full power of my silicon servant, and of Maple, to study it seriously.

Important note: All the results in this paper were gotten by using the Maple package `GEPNER.txt`, available, free of charge, from the url

`http://www.math.rutgers.edu/~zeilberg/tokhniot/GEPNER.txt` .

Sample input and output files may be gotten from the front of this article:

[1] Exactly thirty years ago, on March 31, 1986, my wife Jane and I were at Doron Gepner's 30th birthday party, (in Princeton) that Ida (Doron's wife) organized in their place. Another guest was a colleague of Doron, an Egyptian postdoc, and we pointed out to him that we have the same name, and that it means a "gift", (presumably "God's gift"), to which he retorted " in your cases it seems to be the devil's gifts".

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gepner.html .

A crash course on Permutation and Word Statistics

A permutation statistics is an integer-valued function on the set of permutations. The most famous one is the **number of inversions**, $inv(\pi)$, (that shows up in the definition of the determinant of a square matrix)

$$inv(\pi_1 \dots \pi_n) := \sum_{1 \le i < j \le n} \chi(\pi_i > \pi_j) \quad ,$$

where $\chi(S)$ is 1 or 0, according to whether S is true or false, respectively.

Almost as famous is Major Percy Alexander MacMahon's statistics, $maj(\pi)$, called the "major index"

$$maj(\pi_1 \dots \pi_n) := \sum_{i=1}^{n-1} i\, \chi(\pi_i > \pi_{i+1}) \quad .$$

The generating functions according to *inv* and *maj* are both given by $[n]! := 1(1+q)(1+q+q^2)\cdots(1+q+\dots+q^{n-1}) = (1-q)\dots(1-q^n)/(1-q)^n)$ as proved by Netto and MacMahon respectively. In particular the permutation statistics *inv* and *maj* are *equally distributed*.

Permutations of length n may be viewed as *words* in the "alphabet" $\{1,2,\dots,n\}$ with exactly one occurrence of each letter. The above definitions of *inv* and *maj* make perfect sense when defined on words, of any length, in the same alphabet, where repetitions (and omissions) are welcome.

Let $\mathcal{W}(a_1,\dots,a_n)$ be the set of words in the alphabet $\{1,\dots,n\}$ with a_1 occurrences of 1, a_2 occurrences of 2, …, a_n occurrences of n. MacMahon proved (Theorems 3.6 and 3.7 in [A])

$$\sum_{w \in \mathcal{W}(a_1,\dots,a_n)} q^{inv(w)} = \frac{[a_1+\dots+a_n]!}{[a_1]!\dots[a_n]!} \quad ,$$

$$\sum_{w \in \mathcal{W}(a_1,\dots,a_n)} q^{maj(w)} = \frac{[a_1+\dots+a_n]!}{[a_1]!\dots[a_n]!} \quad ,$$

(where, as mentioned above, $[m]! := (1-q)(1-q^2)\cdots(1-q^m)/(1-q)^m)$.

In particular they are still *equally distributed*, and Dominique Foata ([F]) gave a gorgeous bijective proof.

Asymptotic Normality

Most (but not all) combinatorial statistics (naturally parametrized by one or more integer parameters), for example tossing a (fair or loaded) coin n times and observing the number of Heads, are *asymptotically normal*, that means that (for the sake of simplicity let's only consider the one parameter case), if you call the sequence X_n, figure out its *mean*, μ_n, (usually extremely easy) (aka as average, aka as expectation), call it μ_n, and its *variance*, $m_2(n)$ (also, usually, fairly easy), and define the *standardized* sequence of random variables

$$Z_n := \frac{X_n - \mu_n}{\sqrt{m_2(n)}} \quad ,$$

then the sequence $\{Z_n\}$, converges, in distribution, to the good-old normal distribution (aka Gaussian distribution) whose probability density function is, famously, $e^{-x^2/2}/\sqrt{2\pi}$. A good way to prove this is to discover explicit expressions for the moments (about the mean), $m_r(n)$ (or at least the leading terms), and prove that the *standardized moments*, $m_r(n)/m_2(n)^{r/2}$ tend, as n goes to infinity, to the moments of the standard normal distribution, that equal 0 when r is odd and $1 \cdot 3 \cdots (r-1)$, when r is even. This approach can be often taught to a computer, see [Z1][Z2]. The asymptotic normality of *inv*, for the two-lettered case, was first proved by Mann and Whitney [MW], and for the general case by Persi Diaconis[D] (and reproved in [CJZ]).

Enter Doron Gepner's Statistics

One way to look at an inversion in a word $w_1 \dots w_m$ is as the number of **pairs** of letters $w_i w_j$ with $1 \le i < j \le m$, whose *reduction* is the permutation of length 2, 21, the only **odd** permutation of length 2.

This leads naturally to the analog for three-letter subwords. Given a word $w = w_1 \dots w_m$ (in any alphabet) consider the set of **triples** $1 \le i < j < k \le m$ such that $w_i w_j w_k$ reduces to one of the three odd permutations of length 3, namely one of the members of $\{132, 213, 321\}$.

And indeed, this came up (naturally!), in Doron Gepner's[G] deep work in conformal field theory, and leads to the following definition.

Definition. The Gepner statistics on a word $w = w_1 \dots w_m$ on any finite, totally ordered, alphabet, denoted by $gep(w)$, is defined by

$$gep(w_1 \dots w_m) := \sum_{1 \le i < j < k \le m} \chi(w_i < w_k < w_j \quad OR \quad w_j < w_i < w_k \quad OR \quad w_k < w_j < w_i) \quad .$$

It seems that the Gepner statistics, *gep*, is much harder to study than the inversion number,*inv*. and it is extremely unlikely that a 'nice' (or even 'ugly') explicit formula for the generating function exists. But it is still interesting to find out whether it is asymptotically normal, and if not, to determine the limiting distribution.

Failing an explicit formula for the generating function, the best that we can hope for is an *algorithm* to compute the first few terms of the generating functions for the Gepner statistics. Using the method of [Z1] and [Z2], we can then infer explicit (rigorously proved!) **polynomial** expressions for the first few moments, and try to see what is going on. It is easy to see by a (fully rigorous) *handwaving argument* that each moment is always *some* polynomial, and it is also easy to bound the degree of the r-th moment to be $3r$ (in the cases considered here). The degree of the polynomial expression, in n, of the r-th moment-about-the mean is lower, in fact, in the two cases considered below, it happens to be $2r$. Since a polynomial of degree d is uniquely determined by $d+1$ distinct values, an experimental-yet-rigorous approach would be to find numerical values and then "fit the data".

Due to the complexity of the Gepner statistics (and our finite time) we will only consider the cases of permutations, i.e. $\mathcal{W}(1,\dots,1)$ (with n 1s), and words in the three-letter alphabet $\{1,2,3\}$ with n occurrences of each of the letters 1, 2, and 3, i.e. $\mathcal{W}(n,n,n)$.

First Surprise: The Gepner Statistics on Permutations is NOT Asymptotically Normal

Define the *Gepner polynomials of the first kind*, $G_n(q)$, to be the polynomial

$$G_n(q) := \sum_{\pi \in S_n} q^{gep(\pi)} \quad .$$

Here are the first 8 Gepner polynomials

$$G_1(q) = 1 \quad , \quad G_2(q) = 2 \quad , \quad G_3(q) = 3q+3 \quad , \quad G_4(q) = 4q^4 + 16q^2 + 4 \quad ,$$

$$G_5(q) = 5\,q^{10} + 25\,q^7 + 25\,q^6 + 10\,q^5 + 25\,q^4 + 25\,q^3 + 5 \quad ,$$

$$G_6(q) = 6\,q^{20} + 36\,q^{16} + 72\,q^{14} + 138\,q^{12} + 216\,q^{10} + 138\,q^8 + 72\,q^6 + 36\,q^4 + 6 \quad ,$$

$$G_7(q) = 7\,q^{35} + 49\,q^{30} + 98\,q^{27} + 49\,q^{26} + 98\,q^{25} + 98\,q^{24} + 147\,q^{23} + 441\,q^{22} + 308\,q^{21} + 196\,q^{20}$$
$$+490\,q^{19} + 539\,q^{18} + 539\,q^{17} + 490\,q^{16} + 196\,q^{15} + 308\,q^{14} + 441\,q^{13} + 147\,q^{12} + 98\,q^{11} + 98\,q^{10}$$
$$+49\,q^9 + 98\,q^8 + 49\,q^5 + 7 \quad ,$$

$$G_8(q) = 8\,q^{56} + 64\,q^{50} + 128\,q^{46} + 288\,q^{44} + 128\,q^{42} + 896\,q^{40} + 1344\,q^{38} + 1600\,q^{36} + 3200\,q^{34}$$
$$+4792\,q^{32} + 4352\,q^{30} + 6720\,q^{28} + 4352\,q^{26} + 4792\,q^{24} + 3200\,q^{22} + 1600\,q^{20} + 1344\,q^{18} + 896\,q^{16}$$
$$+128\,q^{14} + 288\,q^{12} + 128\,q^{10} + 64\,q^6 + 8 \quad .$$

By looking at the Taylor expansion around $q=1$, one can get the so-called factorial moments, and from them the usual moments, and from them the moments about the mean (see [Z1]), and Shalosh B. Ekhad found that the average is

$$\mu_1(n) = \frac{1}{12}\,n\,(n-1)\,(n-2) \quad ,$$

(of course, this is obvious, since it must be $\binom{n}{3}/2$ [why?]). Less trivially (but still humanly doable), it found that the variance, $m_2(n)$, is given by the following polynomial expression

$$m_2(n) = \frac{1}{72}\, n^2\,(n-1)\,(n-2) \quad ,$$

in particular, the standard deviation is $O(n^2)$, while the average is $O(n^3)$, so we have, at least, *concentration about the mean.*

Of course, by symmetry, $m_3(n) = 0$, but much less trivially, we have the following **rigorously proved** formula for the fourth moment

$$m_4(n) = \frac{1}{43200}\, n^3\,(n-1)\,(n-2)\left(29\,n^3 - 71\,n^2 - 26\,n - 16\right) \quad .$$

From this it follows that the *standardized fourth moment*, aka as the **kurtosis**, equals

$$\alpha_4(n) = \frac{\frac{1}{43200}\, n^3\,(n-1)\,(n-2)\left(29\,n^3 - 71\,n^2 - 26\,n - 16\right)}{\left(\frac{1}{72}\, n^2\,(n-1)\,(n-2)\right)^2}$$

and taking the limit as $n \to \infty$, we get that the **limiting kurtosis** is

$$\frac{87}{25} = 3.48 \quad ,$$

and **not** 3, so the Gepner statistics is **not** asymptotically normal, and happens to be *leptokurtic* (statisticians' big word for saying that the kurtosis exceeds 3). We have no clue about the limiting distribution, and would love to know it.

The difficulty, from a computational point of view, is that we have no efficient way to get more Gepner polynomials than by very naive brute force. My former PhD student, Brian Nakamura([N]), found more efficient algorithms for related problems, but this needs more work. On the other hand, for the case below, where we look at the Gepner statistics defined on $\mathcal{W}(n,n,n)$, we can do much better than brute force, and squeeze out explicit expressions for more moments.

Second Surprise: The Gepner Statistics on Words in $\{1,2,3\}^*$ is Asymptotically Logistic

Define the *Gepner polynomials of the second kind*, $g_n(q)$, to be the polynomial

$$g_n(q) := \sum_{w \in \mathcal{W}(n,n,n)} q^{gep(w)} \quad .$$

(Recall that $\mathcal{W}(n,n,n)$ is the set of words with n occurrences each of the letters 1, 2, and 3).

Here are the first five Gepner polynomials of the second kind.

$$g_1(q) = 3q + 3 \quad , \quad g_2(q) = 6\,q^8 + 21\,q^6 + 36\,q^4 + 21\,q^2 + 6 \quad ,$$

$$g_3(q) = 9\,q^{27} + 27\,q^{24} + 108\,q^{21} + 264\,q^{18} + 432\,q^{15} + 432\,q^{12} + 264\,q^{9} + 108\,q^{6} + 27\,q^{3} + 9 \quad ,$$

$$g_4(q) = 12\,q^{64} + 36\,q^{60} + 108\,q^{56} + 336\,q^{52} + 870\,q^{48} + 2016\,q^{44} + 4041\,q^{40} + 6252\,q^{36} + 7308\,q^{32} \quad ,$$

$$+6252\,q^{28} + 4041\,q^{24} + 2016\,q^{20} + 870\,q^{16} + 336\,q^{12} + 108\,q^{8} + 36\,q^{4} + 12 \quad ,$$

$$g_5(q) = 15\,q^{125} + 45\,q^{120} + 135\,q^{115} + 330\,q^{110} + 900\,q^{105} + 2115\,q^{100} + 4710\,q^{95} + 10230\,q^{90} + 21195\,q^{85} + 40290\,q^{80}$$

$$+69423\,q^{75} + 102780\,q^{70} + 126210\,q^{65} + 126210\,q^{60} + 102780\,q^{55} + 69423\,q^{50} + 40290\,q^{45} + 21195\,q^{40} + 10230\,q^{35}$$

$$+4710\,q^{30} + 2115\,q^{25} + 900\,q^{20} + 330\,q^{15} + 135\,q^{10} + 45\,q^{5} + 15 \quad .$$

How to get More Gepner Polynomials?

Consider, for motivation, computing the generating function, according to *inv*, of the words in $\mathcal{W}(a_1, a_2, a_3)$, i.e. how can we generate many terms of the polynomials

$$f(a_1, a_2, a_3) := \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} q^{inv(w)} \quad .$$

Suppose that we did not know MacMahon's explicit formula $[a_1 + a_2 + a_3]!/([a_1]![a_2]![a_3]!)$ for it. A natural approach would be to use *Dynamical Programming* and try to express $f(a_1, a_2, a_3)$ in terms of f with smaller arguments.

Note that for $w' \in \mathcal{W}(a_1 - 1, a_2, a_3)$,

$$inv(w'1) = inv(w') + a_2 + a_3 \quad ,$$

since appending a 1 to w' creates a_2 new inversions due to the a_2 2s, and a_3 new inversions due to the a_3 3's.

Similarly, for $w' \in \mathcal{W}(a_1, a_2 - 1, a_3)$,

$$inv(w'2) = inv(w') + a_3 \quad ,$$

since appending a 2 to w' creates a_3 new inversions due to the a_3 3's.

Finally, for $w' \in \mathcal{W}(a_1, a_2, a_3 - 1)$,

$$inv(w'3) = inv(w') \quad ,$$

since appending a 3 to w' does not create any new inversions. Since

$$\mathcal{W}(a_1, a_2, a_3) = \mathcal{W}(a_1 - 1, a_2, a_3)1 \cup \mathcal{W}(a_1, a_2 - 1, a_3)2 \cup \mathcal{W}(a_1, a_2, a_3 - 1)3 \quad ,$$

we have

$$f(a_1,a_2,a_3) := \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} q^{inv(w)}$$

$$= \sum_{w' \in \mathcal{W}(a_1-1,a_2,a_3)} q^{inv(w')+a_2+a_3} + \sum_{w' \in \mathcal{W}(a_1,a_2-1,a_3)} q^{inv(w')+a_3} \sum_{w' \in \mathcal{W}(a_1,a_2,a_3-1)} q^{inv(w')}$$

$$q^{a_2+a_3} f(a_1-1,a_2,a_3) + q^{a_3} f(a_1,a_2-1,a_3) + f(a_1,a_2,a_3-1) \quad .$$

We have just established the recurrence

$$f(a_1,a_2,a_3) = q^{a_2+a_3} f(a_1-1,a_2,a_3) + q^{a_3} f(a_1,a_2-1,a_3) + f(a_1,a_2,a_3-1) \quad ,$$

that would enable us to crank out many $f(n,n,n)$.

In fact, this recurrence can be also used to prove MacMahon's formula, by verifying that $[a_1 + a_2 + a_3]!/([a_1]![a_2]![a_3]!)$ also satisfies the same recurrence (a trivial high-school algebra verification, even for humans), and that the boundary conditions agree.

How can we generalize this argument for computing

$$F(a_1,a_2,a_3) := \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} q^{gep(w)} \quad .$$

Let's use, once again the decomposition

$$\mathcal{W}(a_1,a_2,a_3) = \mathcal{W}(a_1-1,a_2,a_3)1 \cup \mathcal{W}(a_1,a_2-1,a_3)2 \cup \mathcal{W}(a_1,a_2,a_3-1)3 \quad .$$

Let's consider a member, w, of $\mathcal{W}(a_1,a_2,a_3)$, that ends with the letter 1, so $w = w'1$ for $w' \in \mathcal{W}(a_1-1,a_2,a_3)$. Appending 1 to w' introduces new contributions to the *gep* statistics due to all the occurrences of the pairs 32.

Let's consider a member, w, of $\mathcal{W}(a_1,a_2,a_3)$, that ends with the letter 2, so $w = w'2$ for $w' \in \mathcal{W}(a_1,a_2-1,a_3)$. Appending 2 to w' introduces new contributions to the *gep* statistics due to the occurrences of the pairs 13.

Let's consider a member, w, of $\mathcal{W}(a_1,a_2,a_3)$, that ends with the letter 3, so $w = w'3$ for $w' \in \mathcal{W}(a_1,a_2,a_3-1)$. Appending 3 to w' introduces new contributions to the *gep* statistics due to the occurrences of the pairs 21.

This forces us to keep track of the number of occurrences of the pairs 32, 13, and 21.

So we have

$$gep(w'1) - gep(w') = \#32(w') \quad , \quad gep(w'2) - gep(w') = \#13(w') \quad , \quad gep(w'3) - gep(w') = \#21(w') \quad .$$

We also have

$$\#32(w'1) - \#32(w') = 0 \quad , \quad \#32(w'2) - \#32(w') = a_3 \quad , \quad \#32(w'3) - \#32(w') = 0 \quad ,$$

$$\#13(w'1) - \#13(w') = 0 \quad , \quad \#13(w'2) - \#13(w') = 0 \quad , \quad \#13(w'3) - \#13(w') = a_1 \quad ,$$

$$\#21(w'1) - \#21(w') = a_2 \quad , \quad \#21(w'2) - \#21(w') = 0 \quad , \quad \#21(w'3) - \#21(w') = 0 \quad .$$

Defining the family of polynomials in q and the three **catalytic variables** t_{32}, t_{13}, and t_{21}:

$$\mathbf{F}(a_1, a_2, a_3; q, t_{32}, t_{13}, t_{21}) := \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} q^{gep(w)} {t_{32}}^{\#32(w)} {t_{13}}^{\#13(w)} {t_{21}}^{\#21(w)} \quad ,$$

we get the **functional-recurrence** equation

$$\mathbf{F}(a_1, a_2, a_3; q, t_{32}, t_{13}, t_{21}) =$$

$$t_{21}^{a_2} \cdot \mathbf{F}(a_1-1, a_2, a_3; q, qt_{32}, t_{13}, t_{21}) + t_{32}^{a_3} \cdot \mathbf{F}(a_1, a_2-1, a_3; q, t_{32}, qt_{13}, t_{21}) + t_{13}^{a_1} \cdot \mathbf{F}(a_1, a_2, a_3-1; q, t_{32}, t_{13}, qt_{21}) \quad .$$

At the *end of the day*, we plug-in $t_{32} = 1$, $t_{13} = 1$, $t_{21} = 1$, and get our object of desire

$$F(a_1, a_2, a_3; q) = \mathbf{F}(a_1, a_2, a_3; q, 1, 1, 1) \quad .$$

and finally, we can get quite a few Gepner polynomials of the second kind from

$$g_n(q) = F(n, n, n; q) \quad .$$

But we can do better! If we are only interested in, say, the first 20 moments, then we don't need the full polynomials. Since the first r factorial (and hence true) moments can be gotten from the first r Taylor coefficients, about $q = 1$, of $g_n(q)$, we can do a change variables

$$q = 1 + p \quad , \quad t_{32} = 1 + x \quad , \quad t_{13} = 1 + y \quad , \quad t_{21} = 1 + z \quad ,$$

and define

$$\mathbf{H}(a_1, a_2, a_3; p, x, y, z) := \mathbf{F}(a_1, a_2, a_3; 1 + p, 1 + x, 1 + y, 1 + z) \quad .$$

We get a corresponding functional recurrence equation for $\mathbf{H}$, and **now** at each step, we can truncate, and only retain the terms in p, x, y, z of total degree $\leq r$, and of course, at the end of the day, plug-in $x = 0, y = 0, z = 0$. This enabled us to get many more truncated Gepner polynomials, and enabled us to find explicit expressions for the first 12 moments. This leads to

Theorem: The average of the Gepner statistics, defined on the set of words in $\{1, 2, 3\}$ with n occurrences each of 1, 2, and 3, is (of course)

$$\mu_n = \frac{n^3}{2} \quad .$$

The variance is

$$m_2(n) = \frac{n^4}{4} \quad .$$

Of course, all the odd moments are zero. We also have, for the fourth through 12th moments we have the following polynomial expressions.

$$m_4(n) = \frac{1}{80} n^7 \left(-16 + 21\, n\right) \quad .$$

$$m_6(n) = \frac{1}{448} n^9 \left(279\, n^3 - 656\, n^2 + 512\, n - 128\right) \quad .$$

$$m_8(n) = \frac{1}{1280} n^{11} \left(3429\, n^5 - 16480\, n^4 + 32512\, n^3 - 32512\, n^2 + 16384\, n - 3328\right) \quad .$$

$$m_{10}(n) = \frac{1}{11264} n^{13} \left(3\, n - 4\right) \cdot$$
$$\left(68985\, n^6 - 469716\, n^5 + 1391760\, n^4 - 2251584\, n^3 + 2072832\, n^2 - 1022208\, n + 209920\right) \quad .$$

$$m_{12}(n) = n^{15} (\frac{343717911}{1863680} n^9 - \frac{265635477}{116480} n^8$$
$$+ \frac{93771627}{7280} n^7 - \frac{45012561}{1040} n^6 + \frac{43093563}{455} n^5 - \frac{77940021}{560} n^4 + \frac{62196357}{455} n^3 - \frac{39213532}{455} n^2 + \frac{14354176}{455} n - \frac{463661}{91}) \quad .$$

Now let's standardize and take the limit as n goes to infinity. The limiting kurtosis is

$$\kappa_4 = \lim_{n\to\infty} \frac{m_4(n)}{m_2(n)^2} = \frac{21}{5} = 4.2 \quad ,$$

so now the Gepner statistics is even more *leptokurtic*.

The standardized sixth moment tends to

$$\kappa_6 = \lim_{n\to\infty} \frac{m_6(n)}{m_2(n)^3} = \frac{279}{7} \quad .$$

The standardized eighth moment tends to

$$\kappa_8 = \lim_{n\to\infty} \frac{m_8(n)}{m_2(n)^4} = \frac{3429}{5} \quad .$$

Similarly, $\kappa_{10} = 206955/11$ and $\kappa_{12} = 343717911/455$. A quick *google* search for "21/5, 279/7 , 3429/5" revealed that these are the $4th$, $6th$, and $8th$ moments of the standard **Logistic distribution**, whose probability density function is

$$\frac{\pi}{4\sqrt{3}} sech^2(\frac{\pi x}{2\sqrt{3}}) \quad ,$$

and whose moments are $(2^n - 2) \cdot |B_n| \cdot 3^{n/2}$, where B_n are the Bernoulli numbers.

So the first 12 moments perfectly agree, and I am sure that they all do, and hence I can assert with great confidence the interesting

Fact: Doron Gepner's Statistics on the set of words $\mathcal{W}(n,n,n)$ has mean $\frac{n^3}{2}$, variance $\frac{n^4}{4}$ and its limiting (scaled) distribution is the **logistic distribution**.

Let me conclude with

YOM HULEDET SHISHIM SAMEACH, DORON! .

Acknowledgment

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