AUTOMATIC DISCOVERY OF IRRATIONALITY PROOFS AND IRRATIONALITY MEASURES

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Dedicated to Bruce Berndt (b. March 13, 1939), on his $80\frac{3}{4}$-th birthday

Abstract. We illustrate the power of *Experimental Mathematics* and *Symbolic Computation* to suggest irrationality proofs of natural constants, and the determination of their irrationality measures. Sometimes such proofs can be fully automated, but sometimes there is still need for a human touch.

The Maple packages. This article is accompanied by **four** Maple packages:

- `ALLADI.txt`: a Maple package, inspired by the Alladi–Robinson article [1]. It does an automated redux of Theorem 1 in their paper, and extends their results to proving irrationality, as well as finding irrationality measures, of constants of the form $\int_0^1 \frac{\mathrm{d}x}{P(x)}$, where $P(x)$ is a quadratic polynomial with integer coefficients.
- `GAT.txt`: a Maple package that includes the former case, but generalizes it to integrals of the form $\int_0^1 \frac{\mathrm{d}x}{a+x^k}$, where a and k are positive integers.
- `BEUKERS.txt`: a Maple package inspired by Beukers' article [5] for getting $\mathbb{Z}$-linear combinations of 1, $\operatorname{dilog}(a/(a-1)))$, and $\log(a/(a-1))$ that are very small, for integers $a \ge 2$. (*NB*: We follow Maple's convention $\operatorname{dilog}(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ for the dilogarithmic function.)
- `SALIKHOV.txt`: a Maple package that generalizes Salikhov's method [11], to discover and prove linear independence measures of $1, \log(a/(a+1)), \log(b/(b+1))$ for many pairs of integers $b > a \ge 2$, in particular, for $1, \log(a/(a+1))$, $\log((a+1)/(a+2))$ for all $a \ge 1$.

They are available, along with numerous output files, in the form of computer-generated articles, from the following url:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gat.html`,
and its mirrors
`http://wain.mi.ras.ru/gat/` and `http://www.math.ru.nl/~wzudilin/gat/`.
The packages are reusable, as long as Maple exists and at least one of the webpages survives.

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Preface: Apéry's astounding proof (and Berndt's seminar talk) and the Almkvist–Zeilberger algorithm. In 1978, 64-year-old Roger Apéry announced, and sketched an amazing proof that $\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}$ is an irrational number. Some of the details were filled in by Henri Cohen and Don Zagier, and the completed proof was the subject of Alf van der Poorten's classic [9].

One of us (DZ) first learned about this proof from an excellent talk by Bruce Berndt, delivered at the University of Illinois, way back in Fall 1979 (when and where DZ had his third postdoc). It is therefore quite appropriate that we dedicate the present paper to Bruce Berndt, since it deals with irrationality of constants inspired by Apéry's seminal proof, exposited so lucidly by Berndt.

Another *leitmotif* of the present paper is the Almkvist–Zeilberger algorithm. Gert Almkvist and Bruce Berndt co-authored a classic expository paper [2] that won a prestigious *MAA Lester Ford award* in 1988. Since Almkvist (1934–2018) was also a good friend, and long-time collaborator, of the second-named author (WZ), and both authors are good friends and admirers of Berndt, it is more than fitting to dedicate this article to Bruce Berndt.

Apéry's proof of the irrationality of $\zeta(3)$**.** Apéry (see [9]) *pulled out of a hat* two **explicit** sequences of rational numbers. The first sequence consisted only of integers,

$$b_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

now known as the Apéry numbers, while the second one was a sequence of rational numbers,

$$a_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2\,m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

It is reasonably easy to check that $p_n := \operatorname{lcm}(1,\dots,n)^3 a_n$ are integers and, of course, $q_n := \operatorname{lcm}(1,\dots,n)^3 b_n$ are integers. (Thanks to the Prime Number Theorem $\operatorname{lcm}(1,\dots,n)$ is $\Omega(e^n)$, where we use the convenient notation $F(n)=\Omega(c^n)$ meaning $\lim_{n\to\infty}(\log F(n))/n = c$.) Then he came up with a real number

$$\delta = \frac{4\log(1+\sqrt{2})-3}{4\log(1+\sqrt{2})+3} = 0.080259\ldots > 0$$

such that, for some constant C (independent of n),

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| \le \frac{C}{q_n^{1+\delta}}.$$

Once all the claims are verified, it follows that $\zeta(3)$ is irrational. (Indeed, if $\zeta(3)$ would have been rational with denominator c, the left side would have been bounded below by $1/(cq_n)$.) It also follows from the latter estimate that the *irrationality measure* of $\zeta(3)$ (see [9]) is bounded above by $1+1/\delta = 12.417820\ldots$.

Beukers' version. Months later, Frits Beukers [5] gave a much simpler rendition of Apéry's construction by introducing a certain explicit triple integral,

$$I(n) := \int_0^1 \int_0^1 \int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} \right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z},$$

and pointing out that

(i) $I(n)$ is small and can be explicitly bounded: $I(n) = O((\sqrt{2}-1)^{4n})$;
(ii) $I(n) = A(n) - B(n)\zeta(3)$ for certain sequences of rational numbers $A(n)$, $B(n)$, that can be explicitly bounded; and
(iii) $\mathrm{lcm}(1,\ldots,n)^3 A(n)$ and $\mathrm{lcm}(1,\ldots,n)^3 B(n)$ are integers.

The final argument is as in Apéry's proof, and both the irrationality of $\zeta(3)$ and the erstimate $12.417820\ldots$ for its irrationality measure folllow.

Shortly after, Krishna Alladi and M. L. Robinson [1] used one-dimensional analogs to reprove the irrationality of $\log 2$, and established an upper bound for its irrationality measure of 4.63 (subsequently improved, see [13]) by considering the simple integral

$$I(n) := \int_0^1 \left(\frac{x(1-x)}{1+x} \right)^n \frac{\mathrm{d}x}{1+x}.$$

Man muss immer umkehren. Carl Gustav Jacob Jacobi told us that *one must always invert*. Of course, he meant that if you have a complicated looking function like $\int_0^x \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$, its inverse function — in this case $\sin x$ — can be much more user-friendly. We understand Jacobi's quip in a different way. Rather than start with a famous constant, say $\zeta(3)$ or $\log 2$, and wreck our brains trying to find Beukers-like or Alladi–Robinson-like integrals that would produce good diophantine approximations, start with a family of integrals, $I(n)$ say, and see

(i) whether $I(0)$ is a familiar constant, let's call it x;
(ii) whether $I(n)$, for integers $n > 0$, can be written as $A(n) - B(n)x$, with $\{A(n)\}$ and $\{B(n)\}$ sequences of rational numbers;
(iii) whether $I(n)$ has exponential decay, i.e. is 'small';

Since $A(n) - B(n)x$ is so close to 0, the rational $A(n)/B(n)$ is very close to x. Write $A(n)/B(n)$ as $A'(n)/B'(n)$, where now $A'(n)$ and $B'(n)$ are *integers*, and verify

(iv) how close is $A(n)/B(n) = A'(n)/B'(n)$ to x, from a *diophantine* (rather than numerical-analysis) point of view.

We are looking for what we identify as an 'empirical delta', let's call it $\delta(n)$, such that

$$\left| x - \frac{A'(n)}{B'(n)} \right| = \frac{1}{B'(n)^{1+\delta(n)}};$$

in other words, we define

$$\delta(n) := \frac{-\log|x - A'(n)/B'(n)|}{\log B'(n)} - 1.$$

If the values of $\delta(n)$, for $990 \le n \le 1000$ say, are all strictly positive, and safely not *too* close to 0, then we can be assured that there *exists* a proof of irrationality of x together with a corresponding rigorous upper bound for its irrationality measure.

However being sure that a proof *exists* is not the same as having one. It would be nice to

(v) have a fully rigorous proof.

First, try to find one yourself, and you are welcome to get help from your computer, that excels not only in *number-crunching*, but also in *symbol-crunching*, but is still not so good in *idea-crunching* (although it is getting better and better!).

If you are stuck, you can always email an expert number-theorist and offer her or him to collaborate with you on the paper

"Proof of the irrationality of x".

If x was not yet proved to be irrational and, furthermore, is sufficiently famous (e.g. Euler's constant, γ, or Catalan's constant, G), you and your collaborator would become famous too (that what happened to Apéry). If the constant in question is complicated and obscure, it is still publishable, at least in the `arXiv`. If x is already proved to be irrational, and there is currently a proved upper bound for the irrationality measure of x and the implied (rigorous) bound from your sequence $A(n)/B(n)$ is better (i.e. smaller) than the previous one, you (and your expert collaborator) can write a paper

"A new upper bound for the irrationality measure of x",

and you (and your expert collaborator) would be known as the current holders of the world record of the irrationality measure of x, until someone else, in turn, would break your record.

In this paper we follow the methodology outlined and show how, using the amazing Almkvist–Zeilberger algorithm [3] that finds (and at the same time, proves!) a linear recurrence equation with polynomial coefficients for such integrals $I(n)$, one can accomplish the four steps (i)–(iv) (that we call *reconnaissance*) very fast and very efficiently, and sometimes (but not always!) the last — 'rigorous' — step (v) can also be automated.

Warm up: Computerized irrationality proof of $\log 2$. Consider the sequence of definite integrals

$$I(n) := \int_0^1 \left(\frac{x(1-x)}{1+x}\right)^n \frac{\mathrm{d}x}{1+x}.$$

The Almkvist–Zeilberger algorithm [3] produces a linear recurrence equation with polynomial coefficients satisfied by $I(n)$:

$$(n+2)\,I(n) + (-6\,n-9)\,I(n+1) + (n+1)\,I(n+2) = 0.$$

From this we can compute, very fast, many values, and find out the 'empirical deltas'. For example,

$$I(50) = -\frac{18270835389224940244881539949907869989471021543939584297773}{172169139124777594800} \\ + 15310086199495855930932559804210504653 \log 2.$$

This implies the rational approximation to $\log 2$ (by 'pretending' that $I(50)$ is zero) of

$$\frac{18270835389224940244881539949907869989471021543939584297773}{2635924360893339481850468164186010894239049450495548604400},$$

whose empirical delta is

$$0.33269846131126944438\ldots.$$

This is encouraging! But we cannot judge from just one data point. We next find that the 'empirical deltas' for $n = 51$ and $n = 53$ are $0.31992792581569268673\ldots$ and $0.30031107795443952791\ldots$, respectively. To get more confidence, we need to go to higher values of n. The lowest empirical delta between $n = 990$ and $n = 1000$ turns out to be

$$0.28193333613008344616\ldots,$$

that is not as good, but is way above 0. It leads to an upper estimate $1 + 1/\delta(n) \leq 4.5469377751717949058$ for the irrationality measure of $\log 2$. This trend persists, so we can be convinced that the integral $I(n)$ is promising. But this is **not** yet a rigorous proof.

Writing $I(n) = A(n) + B(n) \log 2$, the next step is to (automatically!) find the rate of growth of $A(n)$ and $B(n)$. The original proof in [1] used partial fractions, and the saddle-point method, but thanks to the Poincaré lemma (see [9] and [14] for related use and references) we can do it very fast and automatically.

Note that $A(n)$ and $B(n)$ satisfy the same recurrence. In other words,

$$(n+2)\,A\,(n) + (-6\,n - 9)\,A\,(n+1) + (n+1)\,A\,(n+2) = 0,$$
$$(n+2)\,B\,(n) + (-6\,n - 9)\,B\,(n+1) + (n+1)\,B\,(n+2) = 0.$$

The 'constant-coefficient approximation' of the above recurrence is (taking the leading coefficient in n, that happens to be $n^1 = n$)

$$\bar{A}\,(n) - 6\bar{A}\,(n+1) + \bar{A}\,(n+2) = 0,$$

where $\bar{A}(n)$ is an approximation to $A(n)$ that Poincaré proved has the property that

$$\lim_{n \to \infty} \frac{\log \bar{A}(n)}{\log A(n)} = 1,$$

and similarly for $B(n)$ and $\bar{B}(n)$. The *indicial* equation of this constant-coefficient linear recurrence is

$$1 - 6N + N^2 = 0,$$

whose roots are

$$a = 3 + 2\sqrt{2}, \quad b = 3 - 2\sqrt{2}.$$

It follows that $|A(n)|, |B(n)| = \Omega(a^n)$ and that $|I(n)| = \Omega(b^n)$, since $I(n)$ is obviously the sub-dominant solution, of exponential **decay**. There is only one problem: $A(n)$ is not an integer. The computer can easily check, empirically that $A'(n) := \operatorname{lcm}(1,\dots,n)A(n)$ is an integer for $1 \le n \le 1000$, and then try to prove it in general (or get a little help from a human friend). Then defining additionally

$$B'(n) := \operatorname{lcm}(1,\dots,n)B(n), \quad I'(n) := \operatorname{lcm}(1,\dots,n)I(n),$$

we have that $A'(n), B'(n)$ are integers. As before, $\operatorname{lcm}(1,\dots,n) = \Omega(e^n)$, hence

$$|A'(n)|,\ B'(n) = \Omega(a^n\, e^n), \quad |I'(n)| = \Omega(b^n\, e^n).$$

Since we want

$$|I'(n)| = \frac{1}{\Omega(B'(n)^\delta)},$$

we take

$$\delta = \frac{-\log(be)}{\log(ae)} = -\frac{\log b + 1}{\log a + 1} = \frac{\log\big(3+2\sqrt{2}\big) - 1}{\log\big(3+2\sqrt{2}\big) + 1},$$

leading to the Alladi–Robinson upper bound of $1 + 1/\delta$ that equals

$$2\,\frac{\log\big(3+2\sqrt{2}\big)}{\log\big(3+2\sqrt{2}\big) - 1} = 4.6221008324542313342\dots.$$

This has been improved several times [13], first by Ekaterina Rukhadze [10] (see also [15]); the current record of 3.57455391 is due to Raffaele Marcovecchio [7].

The novelty of our approach is that it can be taught to a computer, and everything, except possibly proving the 'divisibility lemma'—the right denominators of $A(n)$ (that in this case is extremely simple, but in other cases may be quite involved).

Using this method, our computer, Shalosh B. Ekhad, proved *ab initio*, *all by itself* (except the simple divisibility lemma) Theorem 1 of [1]. Note that nowhere did we mention *Legendre polynomials* (they turned, in hindsight, to be unnecessary). Furthermore, our approach is *streamlined*, and the formulation of the theorem is more explicit.

Theorem (Alladi–Robinson [1], but with a more explicit formulation)**.** *Let a and b be positive integers such that $a > (b - e^{-1})^2/4$. Then $\log(1 + b/a)$ is an irrational number with the irrationality measure that is at most*

$$\frac{\log\big(2\,a + b - 2\,\sqrt{a(a+b)}\,\big) - \log\big(2\,a + b + 2\,\sqrt{a(a+b)}\,\big)}{\log\big(2\,a + b - 2\,\sqrt{a(a+b)}\,\big) + 1}$$

Computer-generated proof. See
`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oALLADI1.txt`. □

The above computer-generated paper was produced with the Maple package `ALLADI.txt`. We now mention other articles generated by this Maple package.

• If you want to see computer-generated proofs of irrationality of 89 different constants, and possibly new irrationality measures for each of them, you are welcome to read

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oALLADI2.txt` .
They were 'cherry-picked' from the 'candidate pool' of

$$\int_0^1 \frac{\mathrm{d}x}{a+bx+cx^2} \quad \text{with } 1 \le a,b,c \le 10,\ \gcd(a,b,c)=1,$$

that consisted of 841 'applicants', and naturally we only listed our successes. Of course, all such constants are already proved to be irrational by heavy-artillery theorems (like the Lindemann–Weierstrass theorem), but these theorems do not give explicit bounds for the irrationality measure, and may be *ineffective*.

• Moving right along, the computer-generated article
`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oALLADI3.txt`
gives irrationality proofs, and irrationality measures, to 43 constants, for integrals of the form

$$\int_0^1 \frac{\mathrm{d}x}{a+cx^2}$$

for relatively prime integer pairs a, c in the range $3 \le a, c \le 40$. These are probably subsumed in previous works of Salikhov and his students.

• Even more impressive is
`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oALLADI4.txt` ,
that like the above Alladi–Robinson theorem is true for 'infinitely many' constants, i.e. it is true for symbolic a (subject to a congruence condition).

This theorem may be not new, but the *novelty* is that it was completely *computer-generated*. Let us state the theorem proved in that article.

Theorem. *Let a be a positive integer such that $a \bmod 4$ is 3. Then $\arctan(\sqrt{a})/\sqrt{a}$ is an irrational number, with the irrationality measure at most*

$$\frac{\log\bigl(-a+\sqrt{a(a+1)}\bigr)-\log\bigl(a+\sqrt{a(a+1)}\bigr)}{\log\bigl(-a+\sqrt{a(a+1)}\bigr)-\log\sqrt{a}+1}.$$

The Maple packages `GAT.txt` and `BEUKERS.txt`. The Maple package `GAT.txt` did not produce (so far) anything exciting, so we do not talk about it here, but the readers are welcome to explore it on their own.

Up to this point, all our integrals were one-dimensional. The Maple package `BEUKERS.txt` attempts to tweak Beukers' elegant proof [5] of the irrationality of $\zeta(2)$, by trying to see what happens when you look at the double integral

$$E(n,a) := \int_0^1\int_0^1 \left(\frac{x(1-x)y(1-y)}{1-xy/a}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy/a}.$$

The original case of $a=1$ gave an irrationality proof (and measure) for $\zeta(2) = \pi^2/6$, and indeed $E(0,1)=\zeta(2)$. The recurrence for $E(n,1)$ is second-order (the same one with which Apéry proved the irrationality of $\zeta(2)$). However, things get more complicated for higher a.

Since $E(0,a) = a\,\mathrm{dilog}((a-1)/a)$, one could hope that considering the above double integral would yield irrationality proofs for them. Alas, $E(n,a)$ gets 'contaminated' with $\log((a-1)/a)$, and it turns out that

$$E(n,a) = A(n,a) + B(n,a)\,\mathrm{dilog}((a-1)/a) + C(n,a)\log((a-1)/a),$$

for all n, for **three** sequences of rational numbers $\{A(n,a)\}$, $\{B(n,a)\}$, $\{C(n,a)\}$. This can be proved directly, but it follows immediately from the fact that it is true for $n = 0, 1, 2$, and that $E(n,a)$, and hence $A(n,a), B(n,a), C(n,a)$, all satisfy the third-order linear recurrence equation

$$\begin{aligned}
&- a^3\,(n+1)^2\,(a-1)\,(32\,na + 76\,a - 27\,n - 66)\;E(n,a)\\
&\quad + a^2(512\,a^3n^3 + 2752\,a^3n^2 - 1072\,a^2n^3 + 4800\,a^3n - 5792\,a^2n^2\\
&\qquad + 636\,an^3 + 2736\,a^3 - 10140\,a^2n + 3456\,an^2 - 81\,n^3 - 5796\,a^2\\
&\qquad + 6068\,na - 441\,n^2 + 3472\,a - 768\,n - 432)\,E(n+1,a)\\
&\quad + a(256\,a^2n^3 + 1632\,a^2n^2 - 120\,an^3 + 3376\,a^2n - 780\,an^2 - 81\,n^3\\
&\qquad + 2232\,a^2 - 1670\,na - 522\,n^2 - 1170\,a - 1086\,n - 717)\,E(n+2,a)\\
&\quad + (32\,na + 44\,a - 27\,n - 39)\,(3+n)^2\;E(n+3,a) \;=\; 0.
\end{aligned}$$

This complicated recurrence was obtained using the Apagodu–Zeilberger **multi-variable Almkvist–Zeilberger algorithm** [4].

Using this recurrence we prove that, for every integer $a \ge 2$, there exists a **positive** $\delta = \delta(a)$ and three sequences of integers $C_1(n,a)$, $C_2(n,a)$, $C_3(n,a)$ such that

$$\begin{aligned}
&|C_1(n,a) + C_2(n,a)\,\mathrm{dilog}((a-1)/a) + C_3(n,a)\log((a-1)/a)|\\
&\quad \le \frac{\mathrm{CONSTANT}}{\max(|C_1(n,a)|, |C_2(n,a)|, |C_3(n,a)|)^{\delta(a)}}.
\end{aligned}$$

The explicit expression for $\delta(a)$ is a bit involved and we refer the reader to the computer-generated article
`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oBEUKERS1.txt` ,
that contains a fully rigorous proof of this theorem.

If we did not know that $\log((a-1)/a)$ was irrational, this theorem would have implied that $\mathrm{dilog}((a-1)/a)$ and $\log((a-1)/a))$ can not be both rational. It is not enough, by itself, to prove the linear independence of the three numbers $1, \log((a-1)/a), \mathrm{dilog}((a-1)/a)$ over the rationals, but some human modification of it makes the things work well — see the latest achievements in this direction, together with historical notes, in the wonderful paper [12] of Georges Rhin and Carlo Viola.

The Maple package `SALIKHOV.txt`. We are most proud of this last Maple package, since it generated a **new** theorem that should be of interest to 'mainstream', human mathematicians. It was obtained by generalizing Vladislav Salikhov's proof [11] of the linear independence of $1, \log 2, \log 3$. Our computer proved the following result.

Theorem. *For any integer* $a > 0$, *three numbers* 1, $\log(a/(a{+}1))$, *and* $\log((a{+}1)/(a{+}2))$ *are linearly independent over the rationals. Moreover, there exists a positive number* $\nu(a)$ *such that if* q, p_1, p_2 *are integers and* $Q = \max(|q|, |p_1|, |p_2|) \geq Q_0$, *where* Q_0 *is a sufficiently large number, then*

$$|q + p_1 \log(a/(a+1)) + p_2 \log((a+1)/(a+2))| > Q^{-\nu(a)}.$$

The full proof is in the following article:
`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSALIKHOV2.txt` ,
where an exact expression for $\nu(a)$ can be found (see also below). The theorem was previously known to be true for $a \geq 53$ by Masayoshi Hata [6] (recently improved, though somewhat implicitly, by Volodya Lysov to $a \geq 32$).

Because of the novelty, we choose this result to feature some human-generated highlights of the proof. The integrals

$$E_1(n) = \int_0^{2a+1} \left(\frac{x^2(x^2-(2a+1)^2)(x^2-(2a+3)^2)}{(x^2-(2a+1)^2(2a+3)^2)^2} \right)^n \frac{\mathrm{d}x}{x^2-(2a+1)^2(2a+3)^2}$$

and

$$E_2(n) = \int_0^{2a+3} \left(\frac{x^2(x^2-(2a+1)^2)(x^2-(2a+3)^2)}{(x^2-(2a+1)^2(2a+3)^2)^2} \right)^n \frac{\mathrm{d}x}{x^2-(2a+1)^2(2a+3)^2}$$

are generalizations of integrals in Salikhov's article [11] , and we have

$$E_1(n) = A_1(n) + B(n) \log \frac{a+1}{a+2} \quad \text{and} \quad E_2(n) = A_2(n) + B(n) \log \frac{a}{a+1},$$

where all $E_1(n)$, $E_2(n)$, $A_1(n)$, $A_2(n)$, $B(n)$ satisfy the *same* third order linear recurrence equation with polynomial coefficients, the indicial polynomial of whose 'constant-coefficients recurrence approximation' is

$$\begin{aligned} &1 + (4a^4 + 16a^3 - 11a^2 - 54a - 34)N \\ &\quad - (108a^6 + 648a^5 + 1440a^4 + 1440a^3 + 614a^2 + 76a - 1)N^2 + a(a+2)N^3. \end{aligned}$$

This polynomial has three real roots $C_1(a)$, $C_2(a)$, $C_3(a)$ for $a \geq 1$ located as follows:

$$-\frac{1}{4a^2(a+2)^2} < C_3(a) < 0 < C_2(a) < \frac{1}{27a(a+2)} < 108a^2(a+1)^2 < C_1(a);$$

also $C_2(a) > \frac{1}{4a^2(a+2)^2} > |C_3(a)|$ for $a \geq 2$. Furthermore, choosing $K(a) = a(a+2)$ if a is odd, and $K(a) = (a/2)(a/2+1)$ if a is even, we get numbers $K(a)^n A_1(n)$, $K(a)^n A_2(n)$, and $K(a)^n B(n)$ integral. Then

$$\nu(a) \leq -\frac{\log C_1(a) + \log K(a) + 2}{\log C_2(a) + \log K(a) + 2}$$

for $a \geq 2$, and the same formula for $a = 1$ with $\log C_2(a)$ replaced with $\log |C_3(a)|$. The upper bound for $\nu(a)$ is asymptotically $\frac{3 \log a(a+2)}{\log 27 - 2}$ if a is odd and $\frac{3 \log a(a+2)}{\log 108 - 2}$ if a is even, as $a \to \infty$.

Conclusion. Humans, no matter how smart, can only go so far. Machines, no matter how fast, can also only go so far. The future of mathematics depends on a fruitful *symbiosis* of the strong points of both species, as we hope we demonstrated in this modest tribute to Bruce Berndt.

References

[1] K. Alladi and M. L. Robinson, Legendre polynomials and irrationality, *J. Reine Angew. Math.* **318** (1980), 137–155.

[2] G. Almkvist and B. Berndt, Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, pi and the Ladies Diary, *Amer. Math. Monthly* **95** (1988), 585–608.

[3] G. Almkvist and D. Zeilberger, The method of differentiating under the integral sign, *J. Symbolic Computation* **10** (1990), 571–591; `http://www.math.rutgers.edu/~zeilberg/mamarimY/duis.pdf`.

[4] M. Apagodu and D. Zeilberger, Multi-Variable Zeilberger and Almkvist–Zeilberger Algorithms and the Sharpening of Wilf–Zeilberger Theory, *Adv. Appl. Math.* **37** (2006), 139–152; `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multiZ.html`.

[5] F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* **11** (1979), no. 3, 268–272.

[6] M. Hata, The irrationality of $\log(1+1/q)\log(1-1/q)$, *Trans. Amer. Math. Soc.* **350** (1998), 2311–2327.

[7] R. Marcovecchio, The Rhin–Viola method for $\log 2$, *Acta Arith.* **139** (2009), 147–184.

[8] M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$ (A.K. Peters, 1996); freely available from `https://www.math.upenn.edu/~wilf/AeqB.html`.

[9] A. van der Poorten, A proof that Euler missed… Apéry's proof of the irrationality of $\zeta(3)$, An informal report, *Math. Intelligencer* **1** (1979), 195–203; `http://www.ega-math.narod.ru/Apery1.htm`.

[10] E. A. Rukhadze, A lower bound for the approximation of $\ln 2$ by rational numbers, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* [*Moscow Univ. Math. Bull.*] **42** (1987), no. 6, 25–29 (Russian).

[11] V. Kh. Salikhov, On the irrationality measure of $\ln 3$, *Doklady Math.* **76** (2007), 955–957.

[12] G. Rhin and C. Viola, Linear independence of 1, Li_1 and Li_2, *Moscow J. Combin. Number Theory* **8** (2019), no. 1, 81–96.

[13] E. W. Weisstein, *Irrationality Measure*, from MathWorld — A Wolfram Web Resource; `http://mathworld.wolfram.com/IrrationalityMeasure.html`.

[14] W. Zudilin, Difference equations and the irrationality measure of numbers, *Proc. Steklov Inst. Math.* **218** (1997), 160–174.

[15] W. Zudilin, An essay on irrationality measures of π and other logarithms, *Chebyshevskii Sbornik* **5** (2004), Tula State Pedagogical University, 49–65 (Russian); English version: `http://arxiv.org/abs/math/0404523`.

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