An Empirical Method for Solving (Rigorously!) Algebraic-Functional Equations of the Form $\mathbf{F(P(x,t), P(x,1), x, t) \equiv 0}$

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Added Jan. 2, 2019 (by DZ): Colin R. Defant, from Princeton University, correctly pointed out that there is an issue with the current version. I believe that there is a hidden assumption that the functional equation in question should be solvable in the ring of formal power series. Once this hidden assumption is made explicit, hopefully everything will work out, but right now I am too busy to work out the details.

Preface: It is Never Too Late

One of the references in Doron Zeilberger's article [Z1], written more than 23 years ago, was to an article, labeled *in planning*, with the authors and title of the present article. We must have both forgotten our commitment. But *better late than never*, so with a delay of almost a quarter-century, at long last we got around to writing the promised article. But it is *just as well* that we waited, since now computers are much faster, and we are much better programmers!

Functional Equations

Many enumeration problems reduce to solving a functional equation for a generating function with respect to one or more variables that we **do** care about, and one or more variables, that we **don't** care about, called *catalytic variables*. At the *end of the day* we set all the catalytic variables to 1 (or sometimes 0, and possibly other specific numbers). Even though—at least initially—we do not care about these extra variables (corresponding to auxiliary combinatorial statistics), they are needed to set up the *combinatorial reaction*, so to speak, only to be discarded, like their chemical namesakes, once the 'reaction' is finished.

Initially humans solved these, one at a time, using *ad hoc* human ingenuity, and this goes back to the pioneering work of Tutte and his school on counting maps, in the early 1960s.

More recently, the favorite method became the powerful and sophisticated *kernel method* that had scored many triumphs in the hands of such virtuosi as CNRS Silver-Medalist Mireille Bousquet-Mélou, Marni Mishna, and other people. For lucid and engaging overviews, see the slides of the talks [B1] and [B2]. Alas, this method, in addition to requiring a lot of *human ingenuity*, is also very **human-labor-intensive**.

It turns out, that in *many cases* (perhaps all!), there is an alternative, much simpler, approach, based on empirical *guessing*, **yet it is fully rigorous!** Of course, this method requires the help of our silicon colleagues.

The Zeilberger Gordian Knot

In Zeilberger's proof [Z1] of Julian West's [W] conjectured explicit expression for the number of 2-stack-sortable permutations, it was necessary to solve the functional equation

$$f(x,t) = \frac{1}{1-xt} + \frac{xt(f(x,1)-tf(x,t))(f(x,1)-f(x,t))}{(1-t)^2}. \qquad \text{(FunEq)}$$

By clearing denominators, this equation can be written more compactly as

$$F(\, f(x,t)\, ,\, f(x,1)\, ,\, x\, ,\, t) \;\equiv\; 0,$$

for some polynomial F of four variables.

The idea is extremely simple. Since $f(x,1)$ was conjectured to be a certain known *algebraic* formal power series, why not *guess* that this is also the case for the two-variate formal power series $f(x,t)$; i.e., let the computer **guess** a polynomial in three variables—let's call it G—such that

$$G(f(x,t)\, ,\, x\, ,\, t) \;\equiv\; 0. \qquad \text{(AlgEq)}$$

Once guessed, it is purely routine to prove our guess rigorously. Since both (FunEq) and (AlgEq) have unique formal power series solutions, after we define $g(x,t)$ to be the unique solution of (AlgEq), the verification that

$$F(g(x,t)\, ,\, g(x,1)\, ,\, x\, ,\, t) \;\equiv\; 0,$$

is a routine calculation in the *Schützenberger ansatz* [Z2]. In fact, since we know *a priori* that the left side satisfies *some* algebraic equation, all we need is to bound the degrees, and check that the first few coefficients (in x) are identically 0. Since that's how we got it in the first place, we already know that! Nevertheless, the pedantic purist may want to bound the degrees exactly.

The Gessel Shortcut

Alas, guessing the three-variate polynomial G takes a very long time. After the first draft of [Z1] was written, Ira Gessel made the following observation, reproduced at the very end of the final version of [Z1], that we now reproduce.

"**Epilogue: How The Proof Could Have Been Found**

July 2, 1991: The first proof of any conjecture is seldom the shortest. It turns out that the present proof is no exception. Ira Gessel made the brilliant observation that steps 2–5 can be replaced by the following.

Step $2'$: Conjecture a polynomial I, of two variables, such that $I(P(x),x)) = 0$ where $P(x) = f(x,1)$. To prove it rigorously, we must show that the unique $f(x,t)$ that satisfies

$$F(f(x,t), f(x,1), x, t) \equiv 0$$

is such that $I(f(x,1),x) \equiv 0$. Let's write,

$$\text{(i) } F(f(x,t),Q(x),x,t) \equiv 0, \quad \text{(ii) } f(x,1) = Q(x), \quad \text{(iii) } I(Q(x),x) \equiv 0.$$

We have to prove that (i)+(ii) implies (iii). But note that (i)+(ii) have a unique solution, and (i)+(iii) have a unique solution, and we must show that these are the same. So it's enough to show that (i)+(iii) implies (ii). Taking the resultant of F and I w.r.t. $Q(x)$ gives the algebraic equation $G(f,x,t) \equiv 0$ found empirically, and very painfully, in step 3. Proceeding as in step 5, we see that indeed $Q(x) = P(x)$. This observation is the *leitmotif* of a paper that Ira Gessel and I hope to write."

Implementation

This method turns out to be applicable to many other functional equations. Using Maple, it is easy to follow Gessel's advice. It is very fast to guess a polynomial of two variables, let's call it I, such that

$$I(f(x,1),x) \equiv 0.$$

By the assumption, we have the relation

$$F(f(x,t)\,,\,f(x,1)\,,\,x\,,\,t) \equiv 0.$$

Now eliminate $f(x,1)$ from both equations, using, say, Maple's built-in command `resultant`, and get, an algebraic equation $G(f(x,t),x,t) \equiv 0$, so far only conjectured, linking $f(x,t)$, x, and t, **without** $f(x,1)$. This is *much* faster than direct guessing.

To actually prove that equation, we proceed as above. Defining $g(x,t)$ to be the unique solution of $G(g(x,t),x,t) \equiv 0$, we have to prove that $F(g(x,t)\,,\,g(x,1)\,,\,x\,,\,t)$ is identically 0. This is easily decidable, since we are in the Schützenberger ansatz. One way is to bound the integer N_0 that has the property that if the first N_0 coefficients of the powers of x are 0 then all of them are 0, and then let the computer check these first N_0 coefficients.

Linear Recurrence Equations with Polynomial Coefficients

It is well-known, and easy to see (and implement, e.g., in the Maple package `gfun` described in [SZ]) that once a formal power series satisfies an algebraic equation, as above, it also satisfies a *linear differential equation with polynomial coefficients* (i.e., is D-finite), and hence the enumerating sequence itself, satisfies a *linear recurrence equation with polynomial coefficients* (i.e., is P-recursive). All this can be found automatically, and in fact, since we know *a priori* that such a recurrence is bound to exist, it is completely legitimate to guess it empirically. If we are lucky, and that recurrence happens to be *first-order*, then we get a closed-form 'elegant' expression, like in the case of [W], first proved in [Z1].

The Maple Package FunEq

Everything here is fully implemented in the Maple package `FunEq`, available from the front of this article: `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/funeq.html` . It also contains quite a few sample input and output files that readers are welcome to extend. In particular, we give fully rigorous automatic proofs of all the results in [CJS] (and many other ones, where the answer is not 'nice'), as well as a two-second proof of the main result of [Z1], and we solve numerous other functional equations that we picked more or less at random, just to test the method.

Future Work: More Catalytic Variables; Beyond the Schützenberger Ansatz

The present, naive, guessing approach should be applicable to functional equations with more than one catalytic variable, but then, according to [B1] and [B2], one may have to go to other ansatzes, first D-finite, and then formal power series that satisfy an *algebraic differential equation* rather than a linear one. Sooner or later, things would become too difficult even for computers, but one can do lots of shortcut tricks, and we are sure that the present empirical-yet-rigorous approach should be extendible, and *implementable*. Whether you would get nice results that humans can enjoy remains to be seen, and is rather unlikely. Hence it may not be worth the effort.

References

[B1] M. Bousquet-Mélou, *Enumeration with "catalytic" parameters: a survey*, Three lectures at the 67th Séminaire Lotharingien de Combinatoire, Bertinoro, Italy, September 2011. Available from `http://www.mat.univie.ac.at/~slc/wpapers/s67vortrag/bousquet.pdf` .

[B2] M. Bousquet-Mélou, *Méli-mélo(u) de combinatoire*, exposé pour la remise de la médaille d'argent du CNRS, Bordeaux, Oct. 2, 2014. Available from `http://www.labri.fr/Perso/~bousquet/Exposes/medaille.pdf` .

[CJS] R. Cori, B. Jacquard, and G. Schaeffer, *Description trees for some families of planar maps*, "Proceedings of the 9th Conference on Formal Power Series and Algebraic Combinatorics " (Vienna, 1997), 196–208. Available from `http://www-igm.univ-mlv.fr/~fpsac/FPSAC97/ARTICLES/Schaeffer.ps.gz` .

[SZ] B. Salvy and P. Zimmermann, *GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable*, ACM Trans. Math. Software **20** (1994), 163–177.

[W] J. West, *Sorting twice through a stack*, Theoretical Computer Science **117** (1993), 303–313.

[Z1] D. Zeilberger, *A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length n is* $2(3n)!/((n+1)!\,(2n+1)!)$, Discrete Mathematics **102** (1992), 85–93. Available from `http://www.math.rutgers.edu/~zeilberg/mamarimY/julian.pdf` .

[Z2] D. Zeilberger, *An enquiry concerning human (and computer!) [mathematical] understanding*,

in: "Randomness and Complexity, From Leibniz to Chaitin", Cristian S. Calude, ed., World Scientific, Singapore, 2007, 383-410. Available from
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/enquiry.html` .

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