A BINOMIAL COEFFICIENT IDENTITY ASSOCIATED TO A CONJECTURE OF BEUKERS

Scott Ahlgren, Shalosh B. Ekhad, Ken Ono and Doron Zeilberger

If n is a positive integer, then let $A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$, and define integers $a(n)$ by $\sum_{n=1}^{\infty} a(n)q^n := q \prod_{n=1}^{\infty} (1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - \cdots$. Beukers conjectured that if p is an odd prime, then

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}. \tag{1}$$

In [A-O] it is shown that (1) is implied by the truth of the following identity.

Theorem. *If n is a positive integer, then*

$$\sum_{k=1}^{n} k \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \frac{1}{2k} + \sum_{i=1}^{n+k} \frac{1}{i} + \sum_{i=1}^{n-k} \frac{1}{i} - 2\sum_{i=1}^{k} \frac{1}{i} \right\} = 0.$$

Remark. This identity is easily verified using the WZ method, in a generalized form [Z] that applies when the summand is a hypergeometric term times a WZ potential function. It holds for all positive n, since it holds for n=1,2,3 (check!), and since the sequence defined by the sum satisfies a certain (homog.) third order linear recurrence equation. To find the recurrence, and its proof, download the Maple package EKHAD and the Maple program zeilWZP from `http://www.math.temple.edu/~zeilberg` . Calling the quantity inside the braces $c(n,k)$, compute the WZ pair (F,G), where $F = c(n,k+1) - c(n,k)$ and $G = c(n+1,k) - c(n,k)$. Go into Maple, and type `read zeilWZP; zeilWZP(k*(n+k)!**2/k!**4/(n-k)!**2,F,G,k,n,N):`

Added May 1, 2016: Shalosh B. Ekhad and Doron Zeilberger are now (since 2001) at Rutgers University. The above urls are no longer valid. "temple" should be replaced by "rutgers". The current url for the front of this article, with links to the Maple packages EKHAD and zeilWZP, and the relevant input and outputs files is:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/frits.html` .

The third author is supported by NSF grant DMS-9508976 and NSA grant MSPR-Y012. The last author is supported in part by the NSF.

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References

[A-O] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences* **(in preparation)**.

[B] F. Beukers, *Another congruence for Apéry numbers*, J. Number Th. **25** (1987), 201-210.

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Department of Mathematics, Penn State University, University Park, Pennsylvania 16802
E-mail address: `ahlgren@math.psu.edu`

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: `ekhad@math.temple.edu; http://www.math.temple.edu/ ~ekhad`

Department of Mathematics, Penn State University, University Park, Pennsylvania 16802
E-mail address: `ono@math.psu.edu; http://www.math.psu.edu/ono/`

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: `zeilberg@math.temple.edu; http://www.math.temple.edu/~zeilberg`