THE ENUMERATION OF PERMUTATIONS WITH A PRESCRIBED NUMBER OF "FORBIDDEN" PATTERNS

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Abstract. We initiate a general approach for the fast enumeration of permutations with a prescribed number of occurrences of 'forbidden' patterns, that seems to indicate that the enumerating sequence is always P-recursive. We illustrate the method completely in terms of the patterns 'abc','cab' and 'abcd'.

0. Introduction

The only increasing permutation on $\{1, 2, \ldots, n\}$, $[1, 2, \ldots, n]$, has the property that it has no decreasing subsequence of length 2, i.e. there are no i and j such that $1 \leq i < j \leq n$, and $\pi[i] > \pi[j]$. Thus one measure of how scrambled a permutation is, is its *number of inversions*, $\mathrm{inv}(\pi)$, which is the number of occurrences of the 'pattern' *ba*.

This can be generalized to an arbitrary set of patterns. Given a permutation on $\{1, \ldots, n\}$, we define a *pattern* as a permutation on $\{1, \ldots, r\}$ where $r \leq n$. For $r \leq n$, we say that a permutation $\sigma \in S_n$ has the pattern $\pi \in S_r$ if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $\pi \equiv [\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)]$ in reduced form.

The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_n\}$ where $j_1 < j_2 < \cdots < j_n$ is the permutation $\sigma_1 \in S_n$ obtained by renaming the objects of the permutation σ in the obvious way, so that j_1 is renamed 1 and j_2 is renamed 2 and so on. Thus the reduced form of the permutation 25734 is 14523 and the reduced form of 579 is 123.

To simplify things, when discussing a particular pattern, we will use its alphabetic equivalent. Thus an increasing subsequence of length 3 is an *abc* pattern, an inversion is a *ba* pattern and a decreasing subsequence of length 4 is a *dcba* pattern. Other patterns we will discuss include *abcd*, *bac* and *cab*.

The number of permutations which contain no increasing subsequence of length three (i.e. *abc* avoiding) is known to be $C_n := \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. It is also known [6] that given any pattern of length three, the number of permutations avoiding that pattern is also C_n.

Herb Wilf raised the question: For any pattern π, what can you say about $a_\pi(n)$, the number of permutations on $\{1 \ldots n\}$ that avoid the pattern π?

It follows from the Robinson-Schenstead algorithm and the hook-length formula[3] that for any r, the number of permutations with no increasing subsequence of length r, is a certain binomial-coefficient multisum, from which it follows immediately [8] that it is P-recursive (holonomic) (i.e., it satisfies a linear recurrence with polynomial coefficients in n).

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A natural conjecture is: For any given finite set of patterns, PAT, the sequence

$$a_{PAT}(n) := \left| \left\{ \sigma \in S_n : \ \sigma \text{ has no occurrences of the given patterns} \right\} \right|$$

is P-recursive. More generally, for any such set of patterns $PAT = \{pat_1, pat_2, \ldots, pat_r\}$ and any specified sequence of integers m_p, one for each pattern $p \in PAT$,

$$a_{PAT}^{\{m_1, m_2, \ldots, m_r\}}(n) := \left| \left\{ \sigma \in S_n : \ \sigma \text{ has } \textit{exactly } m_i \text{ occurrences of the pattern } pat_i, \ 1 \leq i \leq r \right\} \right|,$$

is P-recursive.

In this paper, we will present a method for the 'fast' (polynomial time in n) enumeration of such sequences $a(n)$, that seems to support our conjecture that it should always be P-recursive (holonomic) in n.

Until later in the paper, we will consider only one pattern, as the main ideas are already present there.

The natural object is the *generating function:*

$$F_n^{\pi}(q) = \sum_{\sigma \in S_n} q^{\varphi(\sigma)} = \sum c_i(n) q^i$$

where $\varphi(\sigma)$ denotes the number of subsequences, $x_1 x_2 \ldots x_r$, present in σ that reduce to the given pattern π. We will later show how to derive a recursive functional equation, from which it should be possible to extract efficient recurrences for the coefficients $c_i(n)$ of $F_n^{\pi}(q)$ for small i. As we said above, our approach seems to indicate that each $c_i(n)$ is P-recursive in n. It is easy to see that it cannot be also $P-$recursive in i.

Another possibility is to expand the polynomials around $q = 1$, so that we have

$$F_n^{\pi}(q) = \sum_{i=0}^{degree} b_i(n)(q-1)^i.$$

Here $b_0(n)$ is the total number of permutations, $n!$. $b_1(n)$ is the total number of $\pi = x_1 x_2 \ldots x_r$ patterns present in all the permutations of set S_n. The average number of $x_1 x_2 \ldots x_r$ patterns present in the permutations on $\{1 \ldots n\}$ would simply be $\dfrac{b_1(n)}{n!}$. It is easy to see [9] that it is always a polynomial in n, as are all the other coefficients $b_i(n)$, for each fixed i. Thus, we can compute the first few coefficients $b_i(n)$ of $F_n^{\pi}(q+1)$ by brute force. However to get the full $F_n^{\pi}(q)$ we would need the full $F_n^{\pi}(q+1)$, and the coefficients $b_i(n)$, while always polynomials in n, get increasingly complicated as i grows bigger.

Instead we will answer a more general question: How many permutations on $\{1 \ldots n\}$ avoid $\pi = x_1 x_2 \ldots x_r$ *and* $\{\pi_i\}_{i=1}^m$ where $\{\pi_i\}$ is a set of other forbidden patterns for which one or more of the entries in the forbidden subsequence is specified. Among the π_i might be the pattern $ab4$ which simply describes an abc pattern in which the last entry is 4. Another might be a $3b$ pattern which would be a non-inversion in which the first entry is 3.

Once we have such a method, we would also like to compute the number of permutations of a given size which avoid a given set of patterns and compute the number of permutations of a given size which have occurences of patterns from the given set, a prescribed number of times. We may ask how many permutations on $\{1 \ldots n\}$ avoid both abc and cab?; how many avoid abc and have exactly 1 cab?; etc.

1 Counting Permutations with a Presribed Number of *abc* Patterns

1.0 Definitions

Definition 1.1. Given $\sigma \in S_n$, an abc pattern is a sequence i, j, k where $0 < i < j < k \leq n$ and $\sigma(i) < \sigma(j) < \sigma(k)$.

Definition 1.2. $\varphi_{abc}(\sigma)$:= the number of abc patterns of σ.

For example $\varphi_{abc}(4321) = 0$, $\varphi_{abc}(1234) = 4$, and $\varphi_{abc}(2314) = 1$.

Definition 1.3. Given $\sigma \in S_n$, an aj pattern is a sequence i, k where $0 < i < k \leq n$ and $\sigma(i) < \sigma(k) = j$.

For example $\varphi_4(15342) = 2$.

Definition 1.4. $\varphi_j(\sigma)$:= the number of aj patterns of σ.

Definition 1.5. $P^{(r)}(n, I)$ is the number of permutations on $\{1 \dots n\}$ with exactly r abc patterns and no aj patterns for $j \leq I$.

We will use $P(n, I)$ to denote $P^{(0)}(n, I)$, i.e. the number of permutations on $\{1 \dots n\}$ with no abc patterns and no aj patterns for $j \leq I$.

Using the above definitions, the polynomial $F_n^\pi(q)$ described earlier would for this example be defined as

$$F_n^{123}(q) := \sum_{\sigma \in S_n} q^{\varphi_{abc}(\sigma)}.$$

So, e.g., $F_1^{123}(q) = 1$, $F_2^{123}(q) = 2$, $F_3^{123}(q) = 5 + q$, $F_4^{123}(q) = 14 + 6q + 3q^2 + q^4$.

Definition 1.6. For $\sigma \in S_n$, define:

$$wt(\sigma) := q^{\varphi_{abc}(\sigma)} q_2^{\varphi_2(\sigma)} q_3^{\varphi_3(\sigma)} q_4^{\varphi_4(\sigma)} \dots q_n^{\varphi_n(\sigma)}.$$

For example, $wt(2314) = q^1 q_2^0 q_3^1 q_4^3 = q q_3 q_4^3$.

Definition 1.7.

$$P_n(q_2, q_3, \dots, q_n; q) := \sum_{\sigma \in S_n} wt(\sigma).$$

P_n is the 'generalized' form of F_n^{123}. Comparing the two we see that $F_n^{123}(q) = P_n(1, 1 \dots, 1; q)$.

1.1 The functional equation

In order to illustrate the present method, we will treat the simplest non-trivial case, by rederiving the well-known formula for the number of permutations on $\{1 \dots n\}$ with no abc patterns by first obtaining a method to compute the number of permutations on $\{1 \dots n\}$ with no abc patterns and no aj patterns for $j < I$ where I is any integer between 0 and n. We then merely set $I = 0$ and obtain the desired sequence.

In order to find explicit or recursive descriptions for the coefficients of F_n^{123}, we will establish a recursive functional equation for P_n. Let $\sigma \in S_n$, $\sigma(n) = i \neq 1$. Let σ_1 be the permutation on $\{1..i-1, i+1..n\}$ obtained by removing the last entry of σ, i.e.

$$\sigma_1(j) := \sigma(j),\ 1 \leq j \leq n-1.$$

Then

$$wt(\sigma) = wt(\sigma_1) q_i^{i-1} q^{\sum_{j=2}^{i-1} \varphi_j(\sigma_1)},$$

hence

$$wt(\sigma) = q^{\varphi_{abc}(\sigma)} \prod_{j=1}^{n} q_j^{\varphi_j(\sigma)} = q^{abc(\sigma_1)} q_i^{i-1} \prod_{j<i} (qq_j)^{\varphi_j(\sigma_1)} \prod_{j>i} q_j^{\varphi_j(\sigma_1)}. \tag{1}$$

If $\sigma(n) = 1$, then defining σ_1 as above with $i = 1$, we have

$$wt(\sigma) = wt(\sigma_1).$$

Summing over all $\sigma \in S_n$ on the left hand side of (1) is equivalent to summing first over i, then over $\sigma_1 \in S_{n-1}$ on the right. Making the necessary shift in variables to account for the fact that σ_1 above is a permutation on $[1, .., i-1, i+1, .., n]$, we have

$$\sum_{\sigma \in S_n} wt(\sigma) = \sum_{i=1}^{n} \sum_{\substack{\sigma(n)=i \\ \sigma \in S_n}} wt(\sigma) = \sum_{\sigma_1 \in S_{n-1}^{(1)}} wt(\sigma_1)$$
$$+ \sum_{i=2}^{n} \sum_{\sigma_1 \in S_{n-1}^{(i)}} q_i^{i-1} \left[\sum_{\sigma_1 \in S_{n-1}} q^{abc(\sigma_1)} \prod_{j<i} (qq_j)^{\varphi_j(\sigma_1)} \prod_{j>i} q_j^{\varphi_j(\sigma_1)} \right]$$

where $S_{n-1}^{(i)}$ is the set of permutations on $\{1, 2, \ldots, i-1, i+1, \ldots, n-1, n\}$.

Hence

$$P_n(q_2, q_3, \ldots, q_n; q) = P_{n-1}(q_3, q_4, \ldots, q_n; q) + \sum_{i=2}^{n} q_i^{i-1} P_{n-1}(qq_2, qq_3, \ldots, qq_{i-1}, q_{i+1}, \ldots, q_n; q). \tag{2}$$

This recurrence for P_n is the basis for all that follows.

1.2 A RECURRENCE FOR $P(n, I)$

We note that from our definition of P_n, $P_n(\overbrace{0, 0, \ldots, 0}^{I-1}, \overbrace{1, \ldots, 1}^{n-I-1}; 0)$ is the number of permutations on $\{1 \ldots n\}$ with no *abc* patterns and no aj patterns for $j \leq I$. So $P_n(1, 1, \ldots, 1; 0)$ is $a_{123}(n)$, the number of permutations on n elements with no *abc* patterns, and $P_n(\overbrace{0, 0, \ldots, 0}^{I-1}, \overbrace{1, \ldots, 1}^{n-I-1}; 0) = P(n, I)$. We first tackle the question of a recurrence for $a_{123}(n) = P_n(1, 1, \ldots, 1; 0)$. Using (2), we have

$$P_n(1, 1, \ldots, 1; 0) = P_{n-1}(1, 1, \ldots, 1; 0) + \sum_{i=2}^{n} P_{n-1}(\overbrace{0, 0, \ldots, 0}^{i-2}, \overbrace{1, \ldots, 1}^{n-i-1}; 0). \tag{3}$$

It would be too much to hope for a nice clean recurrence on the first try, we seem to have picked up a few uninvited guests, namely the $P_{n-1}(0,0,\ldots,0,1,\ldots,1;0)$. By (2) again,

$$P_n(\overbrace{0,0,\ldots,0}^{I-1},\overbrace{1,\ldots,1}^{n-I-1};0) = P_{n-1}(\overbrace{0,0,\ldots,0}^{I-2},\overbrace{1,\ldots,1}^{n-I-1};0) + \sum_{i=I+1}^{n} P_{n-1}(\overbrace{0,0,\ldots,0}^{i-2},\overbrace{1,\ldots,1}^{n-i-1};0). \tag{4}$$

In terms of $P(n,I)$, (4) becomes

$$P(n,I) = P(n-1,I-1) + \sum_{i=I+1}^{n} P(n-1,i-1) = \sum_{i=I}^{n} P(n-1,i-1). \tag{5}$$

The number of permutations on n elements with no abc patterns is $P_n(1,1,\ldots,1;0) = P(n,1) = a_{123}(n)$. Unfortunately when we try to use (5) to find $P(n,1)$ we are required to define $P(n,0)$. To do so, one needs only look as far as (2). By (2), we have

$$P(n,1) = P(n-1,1) + \sum_{i=2}^{n} P(n-1,i-1).$$

Comparing this with (5) we see that we should define $P(n,0) = P(n,1)$. Using this definition, (5) is valid for $n \geq 1, i > 0$. To complete the scheme, we need some form of initial conditions. These are readily supplied by the observation that the number of permutations on n elements with no abc patterns and no aj patterns $2 \leq j \leq n$ is 1, namely the permutation $[n, n-1, \ldots, 3, 2, 1]$, so $P(n,n) = 1$.

Finally, we may simplify (5) by examining
$P(n,I) - P(n,I+1) = \sum_{i=I}^{n} P(n-1,i-1) - \sum_{i=I+1}^{n} P(n-1,i-1) = P(n-1,I-1)$ which gives us

$$P(n,I) = \begin{cases} 1, & \text{if } n = I \\ P(n,1), & \text{if } I = 0 \\ P(n,I+1) + P(n-1,I-1), & \text{otherwise} \end{cases} \tag{6}$$

Using this recurrence, we may quickly generate a large number of $P(n,I)$.

Table 1

Values of $P(n,I)$

n	I=0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	2	1					
3	5	5	3	1				
4	14	14	9	4	1			
5	42	42	28	14	5	1		
6	132	132	90	48	20	6	1	
7	429	429	297	165	75	27	7	1
8	1430	1430	1001	572	275	110	35	8
9	4862	4862	3432	2002	1001	429	154	44
10	16796	16796	11934	7072	3640	1638	637	208

This enables us to conjecture, and immediately prove (by verifying (6) and the initial conditions), the closed form $P(n,I) = \frac{I+1}{n+1}\binom{2n-I}{n}$, the celebrated ballot numbers[3]. Evaluating at $I=1$ yields yet another proof of the well known fact that the number of permutations on $\{1\ldots n\}$ with no abc patterns, $a_{123}(n)$, equals C_n, the Catalan number. This proof is longer and far less elegant than the combinatorial proofs of [6]. Its only virtue is that it illustrates a *general* method.

1.3 The number of permutations with exactly one abc pattern.

Recently one of us [4] proved that the number of permutations on $\{1\ldots n\}$ with exactly one abc pattern is $\frac{3}{n}\binom{2n}{n+3}$. We will now present an alternative proof using the present method.

The number of permutations with exactly one abc pattern is the constant term of the derivative of the function $F = \sum_{\sigma \in S_n} q^{\varphi_{abc}(\sigma)}$. By differentiating (2) with respect to q, we find a recurrence for $\frac{\partial}{\partial q}P_n$,

$$\frac{\partial}{\partial q}P_n(q_2,\ldots,q_n;q) = \frac{\partial}{\partial q}P_{n-1}(q_3,\ldots,q_n;q) + \sum_{i=2}^{n} q_i^{i-1}\frac{\partial}{\partial q}P_{n-1}(qq_2,qq_3,\ldots,qq_{i-1},q_{i+1},\ldots,q_n;q). \quad (7)$$

Here things begin to get a bit tricky and we will need to employ the chain rule. We will evaluate the right side of the above equation in terms of partial derivatives with respect to the positions that q occupies. For example,

$$\frac{\partial}{\partial q}P_3(qq_2,q_3;q) = q_2\frac{\partial}{\partial q_2}P_3\Bigg|_{\substack{q_2\to qq_2\\ q_3\to q_3\\ q\to q}} + \frac{\partial}{\partial q}P_3\Bigg|_{\substack{q_2\to qq_2\\ q_3\to q_3\\ q\to q}}.$$

The preceding notation means: "first find the partial derivative of the function P_3 then make the necessary substitutions for q_i." Continuing we have

$$\frac{\partial}{\partial q}P_n(q_2,q_3,\ldots,q_n;q) = \frac{\partial}{\partial q}P_{n-1}(q_3,q_4,\ldots,q_n;q)$$
$$+\sum_{i=2}^{n} q_i^{i-1}\left[\frac{\partial}{\partial q}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}} + \sum_{j=2}^{i-1} q_j\frac{\partial}{\partial q_j}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}\right]. \quad (8)$$

We are now ready to tackle $\frac{\partial}{\partial q}P_n\Big|_{\substack{q\to 0\\ q_i\to 1,\ 2\le i\le n}}$

$$\frac{\partial}{\partial q}P_n\Bigg|_{\substack{q_k\to 1,\ 2\le k\le n\\ q\to 0}}$$
$$= \frac{\partial}{\partial q}P_{n-1}\Bigg|_{\substack{q_k\to 1,\ 2\le k\le n-1\\ q\to 0}} + \sum_{i=2}^{n}\left[\frac{\partial}{\partial q}P_{n-1}\Bigg|_{\substack{q_k\to 0,\ 2\le k<i\\ q_k\to 1,\ i\le k\le n-1\\ q\to 0}} + \sum_{j=2}^{i-1}\frac{\partial P_{n-1}}{\partial q_j}\Bigg|_{\substack{q_k\to 0,\ 2\le k<i\\ q_k\to 1,\ i\le k\le n-1\\ q\to 0}}\right].$$

It is readily seen that $P^{(1)}(n,I)$, the number of permutations on $\{1\ldots n\}$ with exactly one abc pattern and no aj patterns for $j\le I$ can be expressed as:

$$P^{(1)}(n,I) = \frac{\partial}{\partial q}P_n\Bigg|_{\substack{q_k\to 0,\ 2\le k<I+1\\ q_k\to 1,\ I+1\le k\le n\\ q\to 0}}.$$

Furthermore, to simplify notation, let

$$P_j^{(1)}(n,I) := \frac{\partial P_n}{\partial q_j}\Bigg|_{\substack{q_k\to 0,\ 2\le k< I+1\\ q_k\to 1,\ I+1\le k\le n\\ q\to 0}}.$$

The actual meaning of $P_j^{(1)}(n,I)$ is immaterial, but the astute reader will see that combinatorially, $P_j^{(1)}(n,I)$ is the number of permutations on $\{1\dots n\}$ with no abc patterns, no ak patterns for $k\le I,\ k\ne j$, and exactly 1 aj pattern if $j\le I$ or *at least* one aj pattern if $j>I$. The recurrence for $P^{(1)}(n,I)$ follows from (8):

$$P^{(1)}(n,I) = P^{(1)}(n-1,I-1) + \sum_{i=I+1}^{n}\left[P^{(1)}(n-1,i-1) + \sum_{j=I+1}^{i-1} P_j^{(1)}(n-1,i-1)\right]. \tag{9}$$

Disregarding for a moment that we do not yet know what $P_j^{(1)}(n,I)$ is, we can state ‘initial’ conditions for this recurrence. First we should define $P^{(1)}(n,0)=P^{(1)}(n,1)$. We can easily compute $P^{(1)}(n,n-2)$. $P^{(1)}(n,n-2)$ is the number of permutations on $\{1\dots n\}$ with exactly 1 abc pattern and no aj patterns for $j\le n-2$. If σ is one such permutation then σ has the form $[n-2,n-1,n-3,\dots,n-i,n,n-i-1,\dots,2,1]$, for some i. There are exactly $n-2$ such permutations, hence $P^{(1)}(n,n-2)=n-2$. Our ‘initial’ conditions (perhaps they should be called ‘boundary’ conditions) for this recurrence are $P^{(1)}(n,0)=P^{(1)}(n,1),\ P^{(1)}(n,n-2)=n-2$.

To obtain a recurrence for $P_j^{(1)}(n,I)$ we must return to (2) and take the partial derivative with respect to q_j. We begin for the case when $j\ge 3$.

$$\begin{aligned}
&\frac{\partial}{\partial q_j}P_n(q_2,\dots,q_n;q)\\
&\quad= \frac{\partial}{\partial q_{j-1}}P_{n-1}\Bigg|_{\substack{q_k\to q_{k+1}\\ q\to q}} + \sum_{i=2}^{j-1} q_i^{i-1}\frac{\partial}{\partial q_{j-1}}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i<k\le n-1\\ q\to q}}\\
&\quad+ \sum_{i=j+1}^{n} qq_i^{i-1}\frac{\partial}{\partial q_j}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i<k\le n-1\\ q\to q}} +(j-1)q_j^{j-2}P_{n-1}(qq_2,\dots,qq_{j-1},q_{j+1},\dots,q_n;q)\ .
\end{aligned}$$

For $j\ge 3$ it follows that

$$P_j^{(1)}(n,I) = P_{j-1}^{(1)}(n-1,I-1) + \sum_{i=I+1}^{j-1} P_{j-1}^{(1)}(n-1,i-1) + (j-1)\chi(j>I)P(n-1,j-1).$$

where $\chi(statement)=\begin{cases}0, & statement \text{ is false}\\ 1, & statement \text{ is true}\end{cases}$. We see that in (9), we only need to know the values of $P_j^{(1)}(n,I)$ for which $j\le I$ so we have

$$P_j^{(1)}(n,I) = P_{(j-1)}^{(1)}(n-1,I-1). \tag{10}$$

When $j=2$ we have

$$\begin{aligned}
P_2^{(1)}(n,I) &= \frac{\partial}{\partial q_2}P_n\Bigg|_{\substack{q_k\to 0,\ 2\le k< I+1\\ q_k\to 1,\ I\le k\le n\\ q\to 0}} = P_{n-1}(\overbrace{0,\dots,0}^{I-2},\overbrace{1,\dots,1}^{n-I};0)\\
&= P(n-1,I-1) \qquad (11)
\end{aligned}$$

Using (11), the recurrence (10) simplifies to

$$\begin{aligned} P_j^{(1)}(n,I) &= P_{j-1}^{(1)}(n-1,I-1) = P_{j-2}^{(1)}(n-2,I-2) = \cdots = P_2^{(1)}(n-j+2,I-j+2) \\ &= P(n-j+1,I-j+1). \end{aligned}$$

Now we can summarize our results so far in a recurrence for $P^{(1)}(n,I)$ (see (9)).

$$P^{(1)}(n,I) = \sum_{i=I}^{n}\left[P^{(1)}(n-1,i-1) + \sum_{j=I+1}^{i-1} P(n-j,i-j)\right]. \tag{12}$$

From (12) it follows that

$$\begin{aligned} P^{(1)}(n,I) - P^{(1)}(n,I+1) &= P^{(1)}(n-1,I-1) + \sum_{i=I}^{n}\left(\sum_{j=I+1}^{i-1} P(n-j,i-j) - \sum_{j=I+2}^{i-1} P(n-j,i-j)\right) \\ &= P^{(1)}(n-1,I-1) + \sum_{i=I+2}^{n} P(n-I-1,i-I-1) \\ &= P^{(1)}(n-1,I-1) + P(n-I,2). \end{aligned}$$

The last equation follows from (5). Finally we have the following recursion for $P^{(1)}(n,I)$:

$$P^{(1)}(n,I) = \begin{cases} P^{(1)}(n,1), & \text{if } I=0 \\ n-2, & \text{if } I=n-2 \\ P^{(1)}(n,I+1) + P^{(1)}(n-1,I-1) + P(n-I,2), & \text{otherwise} \end{cases}$$

Table 2

Values of $P^{(1)}(n,I)$

n	I=0	1	2	3	4	5	6	7	8	9	10
0	0										
1	0	0									
2	0	0	0								
3	1	1	0	0							
4	6	6	2	0	0						
5	27	27	12	3	0	0					
6	110	110	55	19	4	0	0				
7	429	429	229	91	27	5	0	0			
8	1638	1638	912	393	136	36	6	0	0		
9	6188	6188	3549	1614	612	191	46	7	0	0	
10	23256	23256	13636	6447	2601	897	257	57	8	0	0

From this, one conjectures that $P^{(1)}(n,I)$ is given by the expression

$$g(n,I) := \binom{2n-I-1}{n} - \binom{2n-I-1}{n+3} + \binom{2n-2I-2}{n-I-4} - \binom{2n-2I-2}{n-I-1} + \binom{2n-2I-3}{n-I-4} - \binom{2n-2I-3}{n-I-2}.$$

Since it is readily verified that $g(n,I)$ also satisfies the same recurrence and initial conditions, we have a rigorous proof that $P^{(1)}(n,I) = g(n,I)$. Plug in $I=1$ and we find that $a_{123}^{(1)}(n) = g(n,1) = \frac{3}{n}\binom{2n}{n+3}$ as first proved in [4].

1.4 The number of permutations with exactly 2 *abc* patterns

We now will compute $a_{123}^{(2)}$ = the number of permutations in S_n containing exactly two increasing subsequences of length 3. If

$$F_n^{123} = \sum_{\sigma \in S_n} q^{\varphi_{abc}(\sigma)} = a_{123}(n) + a_{123}^{(1)}(n)q + a_{123}^{(2)}(n)q^2 + \dots$$

then

$$\frac{d^2 F_n^{123}}{dq^2} = 2a_{123}^{(2)}(n) + 6a_{123}^{(3)}(n)q + \dots \ .$$

So $a_{123}^{(2)}(n)$ is half the constant term of $\frac{d^2 F_n^{123}}{dq^2}$. Similarly, by our definition of $P^{(r)}(n, I)$ (definition 1.5), we see that

$$P^{(2)}(n, I) = \frac{1}{2}\Phi^{(2)}(n, I).$$

where

$$\Phi^{(2)}(n, I) := \frac{\partial^2}{\partial q^2} P_n \Bigg|_{\substack{q_k \to 0, \ 2 \le k < i \\ q_k \to 1, \ i \le k \le n-1 \\ q \to 0}} .$$

We will find a recursive formula for $\Phi^{(2)}(n, I)$.

From (8), we have

$$\begin{aligned} \frac{\partial^2}{\partial q^2} P_n(q_2, q_3, \dots, q_n; q) &= \frac{\partial^2}{\partial q^2} P_{n-1}(q_3, q_4, \dots, q_n; q) \\ &+ \sum_{i=2}^{n} q_i^{i-1} \left[\frac{\partial^2}{\partial q^2} P_{n-1} \Bigg|_{\substack{q_k \to q q_k, \ 2 \le k < i \\ q_k \to q_{k+1}, \ i \le k < n \\ q \to q}} + \sum_{j=2}^{i-1} q_j \frac{\partial}{\partial q_j} \frac{\partial}{\partial q} P_{n-1} \Bigg|_{\substack{q_k \to q q_k, \ 2 \le k < i \\ q_k \to q_{k+1}, \ i \le k < n \\ q \to q}} \right. \\ &\left. + \sum_{j=2}^{i-1} q_j \left(\frac{\partial}{\partial q} \frac{\partial}{\partial q_j} P_{n-1} \Bigg|_{\substack{q_k \to q q_k, \ 2 \le k < i \\ q_k \to q_{k+1}, \ i \le k < n \\ q \to q}} + \sum_{m=2}^{i-1} q_m \frac{\partial}{\partial q_m} \frac{\partial}{\partial q_j} P_{n-1} \Bigg|_{\substack{q_k \to q q_k, \ 2 \le k < i \\ q_k \to q_{k+1}, \ i \le k < n \\ q \to q}} \right) \right] . \end{aligned} \tag{13}$$

Let $\Phi_{(1,j)}^{(2)}(n, I) = \frac{\partial}{\partial q} \frac{\partial}{\partial q_j} P_n \Bigg|_{\substack{q_k \to 0, \ 2 \le k < i \\ q_k \to 1, \ i \le k \le n-1 \\ q \to 0}}$ and $\Phi_{(j,m)}^{(2)}(n, I) = \frac{\partial^2}{\partial q_j} \frac{\partial}{\partial q_m} P_n \Bigg|_{\substack{q_k \to 0, \ 2 \le k < i \\ q_k \to 1, \ i \le k \le n-1 \\ q \to 0}}$. Then from (13) it follows that

$$\Phi^{(2)}(n, I) = \sum_{i=I}^{n} \left[\Phi^{(2)}(n-1, i-1) + \sum_{j=I+1}^{i-1} \left(2\Phi_{(1,j)}^{(2)}(n-1, i-1) + \sum_{m=I+1}^{i-1} \Phi_{(j,m)}^{(2)}(n-1, i-1) \right) \right] .$$

Subtracting successive terms:

$$\begin{aligned} \Phi^{(2)}(n, I) - \Phi^{(2)}(n, I+1) &= \Phi^{(2)}(n-1, I-1) \\ &+ \sum_{i=I+2}^{n} \left[\Phi_{(I+1,I+1)}^{(2)}(n-1, i-1) + 2\Phi_{(1,I+1)}^{(2)}(n-1, i-1) \right] \\ &+ \sum_{i=I+2}^{n} \sum_{j=I+2}^{i-1} 2\Phi_{(I+1,j)}^{(2)}(n-1, i-1) \end{aligned} \tag{14}$$

We note now that to compute $\Phi^{(2)}(n,I)$ we must also compute $\Phi^{(2)}_{(1,j)}(n,I)$ and $\Phi^{(2)}_{(m,j)}(n,I)$, but we do not need to compute these for all n,I,j,m. We need only compute them for $j,m\le I$. We use (8) to obtain a recursive formula for $\Phi^{(2)}_{(1,j)}(n,I)$, $j\le I$.

$$
\begin{aligned}
&\frac{\partial}{\partial q}\frac{\partial}{\partial q_j}P_n(q_2,q_3,\ldots,q_n;q)\\
&\quad=\sum_{i=2}^{j-1}q_i^{i-1}\left[\sum_{m=2}^{i-1}q_m\frac{\partial}{\partial q_m}\frac{\partial}{\partial q_{j-1}}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}+\frac{\partial}{\partial q}\frac{\partial}{\partial q_{j-1}}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}\right]\\
&\quad+q\sum_{i=j+1}^{n}q_i^{i-1}\left[\sum_{m=2}^{i-1}q_m\frac{\partial}{\partial q_m}\frac{\partial}{\partial q_j}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}+\frac{\partial}{\partial q}\frac{\partial}{\partial q_j}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}\right]\\
&\quad+(j-1)q_j^{j-2}\left[\sum_{i=2}^{j-1}q_i\frac{\partial}{\partial q_i}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<j\\ q_k\to q_{k+1},\ j\le k<n\\ q\to q}}+\frac{\partial}{\partial q}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<j\\ q_k\to q_{k+1},\ j\le k<n\\ q\to q}}\right]\\
&\quad+\sum_{i=j+1}^{n}q_i^{i-1}\frac{\partial}{\partial q_j}P_{n-1}\Bigg|_{\substack{q_k\to qq_k,\ 2\le k<i\\ q_k\to q_{k+1},\ i\le k<n\\ q\to q}}+\frac{\partial}{\partial q}\frac{\partial}{\partial q_{j-1}}P_{n-1}\Bigg|_{\substack{q_k\to q_{k+1}\\ q\to q}}
\end{aligned}
\tag{15}
$$

From this, we have

$$
\Phi^{(2)}_{(1,j)}(n,I)=\begin{cases}\sum_{i=I+1}^{n}P^{(1)}_{(2)}(n-1,i-1)+P^{(1)}(n-1,I-1), & \text{if } j=2\\ \sum_{i=I+1}^{n}P^{(1)}_{(j)}(n-1,i-1)+\Phi^{(2)}_{(1,j-1)}(n-1,I-1), & \text{if } 2<j\le I\end{cases}
$$

where $P^{(1)}_{(j)}(n,I)=\frac{\partial}{\partial q_j}P_n\Big|_{\substack{q_k\to qq_k,\ 2\le k<I\\ q_k\to q_{k+1},\ I\le k<n\\ q\to q}}$. This recurrence can be further simplified using (12) to

$$
\Phi^{(2)}_{(1,j)}=P^{(1)}(n-j+1,I-j+1)+\sum_{k=2}^{j}\left(\sum_{i=I+k-j+1}^{n-j+k}P(n-j,i-k)\right),\ \text{for } j\le I. \tag{16}
$$

We may do the same for $\Phi^{(2)}_{(m,j)}(n,I)$ and obtain

$$
\Phi^{(2)}_{(m,j)}(n,I)=\begin{cases}0, & \text{if } j=m=2\\ 2P(n-1,I-1), & \text{if } j=m=3\le I\\ \Phi^{(2)}_{(j-1,j-1)}(n-1,I-1), & \text{if } j=m\le I\\ P^{(1)}_{(j-1)}(n-1,I-1), & \text{if } j>m=2\\ \Phi^{(2)}_{(m-1,j-1)}(n-1,I-1), & \text{if } j>m,\ m\le I\end{cases}
$$

which collapses down to the 3 cases

$$
\Phi^{(2)}_{(m,j)}(n,I)=\begin{cases}0, & \text{if } j=m=2\\ 2P(n-j+1,I-j+1), & \text{if } j=m\le I\\ P^{(1)}_{(j-m+1)}(n-m+1,I-m+1), & \text{if } j>m,\ m\le I\end{cases}. \tag{17}
$$

One might think that we have overlooked a few cases here, for what if $j > m > I$ or $j = m > I$? Examining (14) it is apparent that to compute $\Phi^{(2)}(n, I)$, we need only compute $\Phi^{(2)}_{(m,j)}$ and $\Phi^{(2)}_{(1,j)}$ when $j, m \leq I$. We now use (5),(12),(16) and (17) to simplify (14):

$$\Phi^{(2)}(n,I) = \Phi^{(2)}(n-1,I-1) + \Phi^{(2)}(n,I+1)$$
$$+ 2P^{(1)}(n-I,2) + 2IP(n-I,3) + 2P(n-i-1,2) + \chi(I>1)2P(n-I+1,3)$$

We may now make the substitution $P^{(2)}(n,I) = \frac{1}{2}\Phi^{(2)}(n,I)$. Our recurrence for $P^{(2)}(n,I)$ can be stated as

$$P^{(2)}(n,I) = \begin{cases} P^{(2)}(n,1), & \text{if } I = 0 \\ n-3, & \text{if } I = n-2 \\ P^{(2)}(n-1,I-1) + P^{(2)}(n,I+1) + P^{(1)}(n-I,2) & \\ \quad +P(n-i-1,2) + IP(n-I,3) + \chi(I>1)P(n-I+1,3), & \text{if } 0 < I < n-2, n > 3 \end{cases}.$$

Table 3

Values of $P^{(2)}(n,I)$

n	I=0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0
4	3	3	1	0	0	0	0	0	0	0	0
5	24	24	12	2	0	0	0	0	0	0	0
6	133	133	74	23	3	0	0	0	0	0	0
7	635	635	371	141	36	4	0	0	0	0	0
8	2807	2807	1688	709	227	51	5	0	0	0	0
9	11864	11864	7276	3248	1168	334	68	6	0	0	0
10	48756	48756	30340	14121	5459	1771	464	87	7	0	0

Using `ordi` of [10] or `gfun`[5] we conjecture that

$$a^{(2)}_{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)} \binom{2n}{n-4}.$$

It is very likely that one should be able to conjecture an explicit expression for $P^{(2)}(n,I)$, which would be routine to prove, and from which the above conjecture would follow.

2 The Method

We may now outline the method described in this paper.

To determine the number of permutations on $\{1 \ldots n\}$ having exactly r occurrences of the pattern $x_1 x_2 \ldots x_k$,

1. *Determine the best way to obtain a recurrence for this pattern.*

There are basically four ways to do this.

a) by removing the last entry of the permutation, as in the *abc* example.

b) by removing the first entry of the permutation, which is what we will do in the next example.

c) by removing n from the permutation.

d) by removing 1 from the permutation.

2. *Identify the other parameters needed in order to describe the recurrence.*

In our first example, we found a recurrence for the number of permutations with a given number of *abc* patterns by looking at what happened to a permutation from which we removed its last object. As a consequence, we were forced to consider the number of *abc* patterns present in each permutation, but also the number of *aj* patterns present. Note that we could arrive at this requirement by noting the result of removing the last object from the pattern *abc*. *abc* becomes *ab*. Since only some choices of b result in a true *abc* pattern for a given c, we must be specific, and count the number of *aj* patterns for every j. We count the number of *aj* by using the parameter q_j in our weight function.

3. *Define* $P_n = \sum_{\sigma \in S_n} wt(\sigma)$.

Here $wt(\sigma) = q^{\varphi}(\sigma) \prod q_j^{\varphi_j(\sigma)}$ where $\varphi_j(\sigma)$ is the number of times the pattern associated with q_j can be found in σ and $\varphi(\sigma)$ is the number of $x_1x_2 \dots x_r$ in σ.

4. *Determine the functional equation.*

Using the recurrence described in **1**, determine a recurrence in n for P_n.

5. *Take the* r^{th} *derivative of the functional equation with respect to* q.

6. *Let* $q = 0$, *and* $q_j = 1$ *for all other parameters of* P_n.

For all intents and purposes, you are now done, for you have obtained a (perhaps complicated) recurrence for the number of permutations on $\{1 \dots n\}$ containing exactly k $x_1x_2 \dots x_r$'s. This recurrence probably involves other terms, but each of these have their own recurrences which can be determined from the functional equation. After all is said and done, you frequently are able to simplify this recurrence, eliminating many of these unwanted terms, but for now, you have achieved your goal.

3 The forbidden pattern *cab*

3.1 Definitions

As with each of the examples we examine, the definitions for P, $P^{(k)}$, φ_j and wt should be taken as being local to the problem at hand.

Definition 3.1. Given $\sigma \in S_n$, a *cab* pattern is a sequence i, j, k where $1 \leq i < j < k \leq n$ and $\sigma(j) < \sigma(k) < \sigma(i)$.

Definition 3.2. For $\sigma \in S_n$, let $\varphi_{cab}(\sigma)$ be the number of cab patterns of σ.

The number of permutations on $\{1 \dots n\}$ having no *cab* patterns will be the constant term of the polynomial $F = \sum_{\sigma \in S_n} q^{\varphi_{cab}(\sigma)}$. As we saw earlier, we may add any number of parameters to this polynomial, as long as we know what to do with them to obtain the constant term. In this case, we will use the parameters $q_2, \dots, q_n$ as we did with *abc*.

Definition 3.3. Given $\sigma \in S_n$,

$$wt(\sigma) = q^{\varphi_{cab}(\sigma)} \prod_{j=2}^{n} q^{\varphi_j(\sigma)}.$$

3.2 No *cab*'s

Let $\sigma(n) = i$. A functional equation results from examining $\sigma_1(k) = \sigma(k),\ 1 \le k < n$,

$$wt(\sigma) = wt(\sigma_1) q^{\sum_{j<i} \varphi_j(\sigma_1)} \prod_{j>i} q_j$$

Summing over all $\sigma \in S_n$ we have

$$P_n(q, q_2, q_3, \dots, q_n) = \sum_{i=2}^{n} \left[\left(\prod_{j>i} q_j \right) P_{n-1}(q, qq_2, \dots, qq_{i-1}, q_{i+1}, \dots, q_n) \right] + \left(\prod_{j} q_j \right) P_{n-1}(q, q_3, \dots, q_n). \tag{18}$$

Not surprisingly, if we let $P(n, I) = P_n(0; \overbrace{0, 0, \dots, 0}^{I-1}, \overbrace{1, \dots, 1}^{n-I-1})$, then we obtain the following recurrence for $P(n, I)$,

$$P(n, I) = \begin{cases} 1 & , \text{ if } n = I \\ P(n, 1) & , \text{ if } I = 0 \\ P(n, I+1) + P(n-1, I-1), & \text{otherwise} \end{cases}$$

Which is identical to (6). So we have reproved that the number of permutations on $\{1 \dots n\}$ with no *abc* subsequences is equal to the number with no *cab* subsequences.

3.3 Permutations with one *cab*

Though the number of permutations with no *abc*'s is equal to the number with no *cab*'s, we will find that this is not the case when we examinee the number of permutations with one *cab*. we follow the same procedure outlined for the *abc* case, with the goal of finding $\left.\frac{\partial}{\partial q} P_n \right|_{\substack{q \to 0 \\ q_j \to 1,\ 1 \le j \le n}}$. Taking derivatives of both sides of (18) with respect to q, we have

$$\frac{\partial}{\partial q} P_n = \sum_{i=1}^{n} \left(\prod_{j>i} q_j \right) \left[\sum_{j=2}^{i-1} q_j \frac{\partial}{\partial q_j} P_{n-1} \right|_{\substack{q_k \to qq_k,\ 2 \le k < i \\ q_k \to q_{k+1},\ i \le k < n \\ q \to q}} + \left. \frac{\partial}{\partial q} P_{n-1} \right|_{\substack{q_k \to qq_k,\ 2 \le k < i \\ q_k \to q_{k+1},\ i \le k < n \\ q \to q}} \Bigg]$$
$$+ \left(\prod_{j} q_j \right) \left. \frac{\partial}{\partial q} P_{n-1} \right|_{\substack{q \to q \\ q_k \to q_{k+1},\ 1 \le k \le n-1}}.$$

We do the same for the derivative with respect to q_j:

$$\frac{\partial}{\partial q_j} P_n = \sum_{i=2}^{j-1} \left(\prod_{\substack{k>i \\ k \neq j}} q_k \right) P_{n-1} \Bigg|_{\substack{q_k \to q q_k,\ 2 \leq k < i \\ q_k \to q_{k+1},\ i \leq k < n \\ q \to q}} + \sum_{i=1}^{j-1} \left(\prod_{k>i} q_k \right) \frac{\partial}{\partial q_{j-1}} P_{n-1} \Bigg|_{\substack{q_k \to q q_k,\ 2 \leq k < i \\ q_k \to q_{k+1},\ i \leq k < n \\ q \to q}}$$

$$+ \sum_{i=j+1}^{n} \left(\prod_{k>i} q_k \right) q \frac{\partial}{\partial q_j} P_{n-1} \Bigg|_{\substack{q_k \to q q_k,\ 2 \leq k < i \\ q_k \to q_{k+1},\ i \leq k < n \\ q \to q}} .$$

Here we let $P^{(1)}(n,I) = \frac{\partial}{\partial q} P_n \Bigg|_{\substack{q \to 0 \\ q_k \to 0,\ k \leq I \\ q_k \to 1,\ k > I}}$ and $P^{(1)}_{(j)}(n,I) = \frac{\partial}{\partial q_j} P_n \Bigg|_{\substack{q \to 0 \\ q_k \to 0,\ k \leq I \\ q_k \to 1,\ k > I}}$. We obtain the following recursions:

$$P^{(1)}_{(j)}(n,I) = \begin{cases} 0, & \text{if } n - I < 0 \text{ } or \text{ } j < I \\ P^{(1)}_{(j)}(n,1), & \text{if } I = 0 \\ P(n-1, I-1), & \text{if } j = I \\ \sum_{k=i}^{j-1} P(n-1, k-1), & \text{if } j > I \end{cases},$$

$$P^{(1)}(n,I) = \begin{cases} I, & \text{if } I = n-2 \\ P^{(1)}(n,1), & \text{if } I = 0 \\ \sum_{i=I}^{n} \left[\sum_{j=I+1}^{i-1} P^{(1)}_{(j)}(n-1, i-1) + P^{(1)}(n-1, i-1) \right], & \text{otherwise} \end{cases}.$$

We combine these two recurrences and localize to obtain

$$P^{(1)}(n,I) = \begin{cases} I, & \text{if } I = n-2 \\ P^{(1)}(n,1), & \text{if } I = 0 \\ P^{(1)}(n, I+1) + P^{(1)}(n-1, I-1) + P(n-2, I), & \text{otherwise} \end{cases}.$$

Table 4

Values of $P^{(1)}(n,I)$

n	I=0	1	2	3	4	5	6	7	8	9	10
0	0										
1	0	0									
2	0	0	0								
3	1	1	0	0							
4	5	5	2	0	0						
5	21	21	11	3	0	0					
6	84	84	49	19	4	0	0				
7	330	330	204	92	29	5	0	0			
8	1287	1287	825	405	153	41	6	0	0		
9	5005	5005	3289	1705	715	235	55	7	0	0	
10	19448	19448	13013	7007	3146	1166	341	71	8	0	0

Using `ordi`[10] or `gfun`[5] we conjecture that $a^{(1)}_{312}(n)$, the number of permutations on $\{1, 2, \dots, n\}$ containing exactly one *cab* subsequence, $= \frac{n-2}{2n}\binom{2n-2}{n-1}$.

4 Counting *abcd*'s

Lest the reader think that the methods outlined in this paper will only help us gain information about permutations avoiding forbidden patterns of length three, here we examine the forbidden pattern *abcd*.

4.1 Definitions

Definition 4.1. An *abcd* pattern of a permutation σ is a sequence $1 \le i < j < k < l \le n$ with $\sigma(i) < \sigma(j) < \sigma(k) < \sigma(l)$.

Definition 4.2. $\varphi_{abcd}(\sigma) =$ the number of *abcd* patterns which can be found in σ.

As before, we have

Definition 4.3. $\varphi_{aj}(\sigma) =$ the number of *aj* patterns of σ.

Here we must also define the following.

Definition 4.4. An *abj* pattern is a sequence $1 \le i_1 < i_2 < i_3 \le n$ with $\sigma(i_1) < \sigma(i_2) < \sigma(i_3) = j$.

Definition 4.5. $\varphi_{abj}(\sigma) =$ the number of *abj* patterns of σ.

Definition 4.6. Let $\sigma \in S_n$ then

$$wt(\sigma) = q^{\varphi_{abcd}(\sigma)} \prod_{j=3}^{n} q_j^{\varphi_{abj}(\sigma)} \prod_{j=2}^{n} \xi_j^{\varphi_{aj}(\sigma)}.$$

The number of permutations with no *abcd*

Let $\sigma_1(j) = \sigma(j),\ 1 \le j \le n-1$. If $\sigma(n) = i$ then

$$wt(\sigma) = wt(\sigma_1) q^{\sum_{j=3}^{i-1} \varphi_{abj}(\sigma_1)} q_i^{\sum_{j=2}^{i-1} \varphi_{aj}(\sigma_1)} \xi_i^{i-1}. \tag{19}$$

Let $P_n(q; q_3, \dots, q_n; \xi_2, \dots, \xi_n) = \sum_{\sigma \in S_n} wt(\sigma)$. Using (19), we have

$$\begin{aligned} P_n(q; q_3, \dots, q_n; \xi_2, \dots, \xi_n) = \sum_{i=3}^{n} \xi_i^{i-1} P_{n-1}(q; qq_3, \dots, qq_{i-1}, q_{i+1}, \dots, q_n; q_i\xi_2, \dots, q_i\xi_{i-1}, \xi_{i+1}, \dots, \xi_n) \\ + \xi_2 P_{n-1}(q; q_4, \dots, q_n; \xi_3, \dots, \xi_n) + P_{n-1}(q; q_4, \dots, q_n; \xi_3, \dots, \xi_n). \end{aligned} \tag{20}$$

The number of permutations on $\{1 \dots n\}$ with no *abcd* patterns is

$$P_n(0; 1, \dots, 1; 1, \dots, 1) = \sum_{i=3}^{n} P_{n-1}(0; \overbrace{0, \dots, 0}^{i-2}, \overbrace{1, \dots, 1}^{n-i-1}; 1, \dots, 1) + 2P_{n-1}(0; 1, \dots, 1; 1, \dots, 1).$$

We may use (20) twice more to find

$$\begin{aligned} P_n(1; \overbrace{0, \dots, 0}^{I_1-2}, \overbrace{1, \dots, 1}^{n-I_1-1}; 1, \dots, 1) = \sum_{i=3}^{I_1} P_{n-1}(0; \overbrace{0, \dots, 0}^{I_1-3}, \overbrace{1, \dots, 1}^{n-I_1-1}; \overbrace{0, \dots, 0}^{i-2}, \overbrace{1, \dots, 1}^{n-i-1}) \\ + \sum_{i=I_1+1}^{n} P_{n-1}(0; \overbrace{0, \dots, 0}^{i-3}, \overbrace{1, \dots, 1}^{n-i-1}; 1, \dots, 1) \\ + 2P_{n-1}(0; \overbrace{0, \dots, 0}^{I_1-3}, \overbrace{1, \dots, 1}^{n-I_1-1}; 1, \dots, 1) \end{aligned}$$

and

$$\begin{aligned} P_n(1;\overbrace{0,\ldots,0}^{I_1-2},\overbrace{1,\ldots,1}^{n-I_1-1};\overbrace{0,\ldots,0}^{I_2-1},\overbrace{1,\ldots,1}^{n-I_2-1}) &= \sum_{i=I_2+1}^{I_1} P_{n-1}(0;\overbrace{0,\ldots,0}^{I_1-3},\overbrace{1,\ldots,1}^{n-I_1-1};\overbrace{0,\ldots,0}^{i-2},\overbrace{1,\ldots,1}^{n-i-1}) \\ &+ \sum_{i=I_1}^{n} P_{n-1}(0;\overbrace{0,\ldots,0}^{i-3},\overbrace{1,\ldots,1}^{n-i-1};\overbrace{0,\ldots,0}^{I_2-2},\overbrace{1,\ldots,1}^{n-I_2-1}) \\ &+ P_{n-1}(0;\overbrace{0,\ldots,0}^{I_1-3},\overbrace{1,\ldots,1}^{n-I_1-1};\overbrace{0,\ldots,0}^{I_2-2},\overbrace{1,\ldots,1}^{n-I_2-1}). \end{aligned} \tag{21}$$

Let $P(n, I_1, I_2) = P_n(1;\overbrace{0,\ldots,0}^{I_1-2},\overbrace{1,\ldots,1}^{n-I_1-1};\overbrace{0,\ldots,0}^{I_2-1},\overbrace{1,\ldots,1}^{n-I_2-1})$ for $0 < \{I_1, I_2\} \le n$. Observe that $P(n,n,n)$ is the number of permutations on $\{1\ldots n\}$ with no *abcd* pattern, no *abc* pattern (for any c) and no aj patterns for any j. There is only one such permutation, namely $[n, n-1, \ldots, 2, 1]$, thus $P(n,n,n) = 1$ for all n. Furthermore, it is clear from (21) that when $I_1 < I_2$, $P(n, I_1, I_2) = P(n, I_2, I_2)$. We may define $P(n, I_1, 0) = P(n, I_1, 1)$ and $P(n, 0, I2) = P(n, 1, I2)$ for all values of n, I_1 and I_2. Using this notation, we have

$$P(n, I_1, I_2) = \begin{cases} P(n, 1, I_2), & \text{if } I_1 = 0 \\ P(n, I_1, 1), & \text{if } I_2 = 0 \\ P(n, I_2, I_2), & \text{if } I_1 < I_2 \\ 1, & \text{if } I_1 = I_2 = n \\ \begin{aligned} \sum_{i=I_2+1}^{I_1} P(n-1, I_1-1, i-1) + \sum_{i=I_1+1}^{n} P(n-1, i-1, I_2) \\ + P(n-1, I_1-1, I_2-1) \end{aligned}, & \text{otherwise} \end{cases}.$$

Table 5

Values of $P(n, I, 1)$

n	I=0	1	2	3	4	5	6	7	8	9	10
0	0										
1	1	1									
2	2	2	2								
3	6	6	6	5							
4	23	23	23	20	14						
5	103	103	103	92	70	42					
6	513	513	513	466	372	252	132				
7	2761	2761	2761	2536	2086	1509	924	429			
8	15767	15767	15767	14594	12248	9227	6127	3432	1430		
9	94359	94359	94359	87830	74772	57894	40403	24882	12870	4862	
10	586590	586590	586590	548325	471795	372565	268909	175474	101036	48620	16796

Table 6

Values of $P(n,1,I)$

n	I=0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	2	2	1								
3	6	6	3	1							
4	23	23	12	4	1						
5	103	103	56	20	5	1					
6	513	513	288	110	30	6	1				
7	2761	2761	1588	640	190	42	7	1			
8	15767	15767	9238	3882	1235	301	56	8	1		
9	94359	94359	56094	24358	8187	2163	448	72	9	1	
10	586590	586590	352795	157265	55235	15575	3528	636	90	10	1

Using `ordi`[10], we conjecture that $a^{(0)}_{1234}(n) = P(n,1,1)$ satisfies the following recurrence with $a^{(0)}_{1234}(0) = 1$ and $a^{(0)}_{1234}(1) = 1$:

$$a^{(0)}_{1234}(n+2) = -9\frac{(n+1)^2}{(n+4)^2}a^{(0)}_{1234}(n) + \frac{10n^2+42n+41}{(n+4)^2}a^{(0)}_{1234}(n+1)$$

5 Counting permutations avoiding more than one forbidden pattern

The method outlined in this paper can be used to find recurrences for the number of permutations avoiding 2 or more forbidden patterns. The method is essentially the same, though you have to keep track of more parameters, and recurrences can be more complicated. In the example that follows, we seek the number of permutations on $\{1\ldots n\}$ avoiding both abc and bac. Let $\varphi_{bac}(\sigma)$ be the number of bac patterns of the permutation σ. Let $\varphi_{ja}(\sigma)$ be the number of ja patterns of σ (that is the number of inversions 'caused' by j). Let

$$wt(\sigma) = q^{\varphi_{abc}(\sigma)}\xi^{\varphi_{bac}(\sigma)}\prod_{j=2}^{n}\left(q_j^{\varphi_{aj}(\sigma)}\xi_j^{\varphi_{ja}(\sigma)}\right).$$

It may seem a bit wasteful to spend time defining both $\varphi_{aj}(\sigma)$ and $\varphi_{ja}(\sigma)$ when it is clear that $\varphi_{aj}(\sigma) + \varphi_{ja}(\sigma) = n-1$ but we will see that this 'complication' in addition to the introduction of the parameters ξ_j will pay off in the end.

$$P_n(q;\xi;q_2,\ldots,q_n;\xi_2,\ldots,\xi_n) = \sum_{\sigma\in S_n} wt(\sigma) = \sum_{i=1}^{n}\sum_{\substack{\sigma\in S_n\\ \sigma(n)=i}} wt(\sigma)$$

$$= \sum_{i=2}^{n} q_i^{i-1}\left(\prod_{j>i}\xi_j\right)P_{n-1}(q;\xi;qq_2,\ldots,qq_{i-1},q_{i+1},\ldots,q_n;\xi\xi_2,\ldots,\xi\xi_{i-1},\xi_{i+1},\ldots,\xi_n)$$

$$+\left(\prod_{j}\xi_j\right)P_{n-1}(q;\xi;q_3,\ldots,q_n;\xi_3,\ldots,\xi_n).$$

Let $P(n,I) = P_n(0;0;\overbrace{0,\dots,0}^{I-1},\overbrace{1,\dots,1}^{n-I};\overbrace{0,\dots,0}^{I-1},\overbrace{1,\dots,1}^{n-I})$. Note that from our definition, $P(n,I) = 0$ when $I > 1$. Indeed, $P(n,I)$ is the number of permutations on $\{1 \dots n\}$ containing no abc, no bac, no aj for $j \le I$ and no ja for $j \le I$. If there was a permutation for which this was true for an I¿1 then we merely examinee the positions of the object 1 and 2 in the permutation. Let $\sigma(i) = 1$ and $\sigma(j) = 2$. If $i < j$ then σ has an $a2$ pattern. If $j < i$ then σ has a $2a$ pattern. Thus, σ does not meet the requirements and so $P(n,I) = 0$ for $I > 1$. We have the recurrence

$$P(n,I) = \begin{cases} P(n,1), & \text{if } I = 0 \\ 0, & \text{if } I > 1 \\ 1, & \text{if } n = I = 1 \\ \sum_{i=I}^{n} P(n-1, i-1), & \text{otherwise} \end{cases}.$$

This recurrence reduces to

$$P(n,1) = \begin{cases} 1, & \text{if } n = 1 \\ 2P(n-1,1), & \text{if } n > 1 \end{cases}.$$

So we have reproved the well know result that the number of permutations on $\{1 \dots n\}$ with no abc and no bac is 2^n.

Conclusion

We saw in the above examples that in order to compute the quantity of interest, a_n, say, we naturally introduced extra parameters $I_1, I_2, \dots$, and a new quantity $F(n, I_1, I_2, \dots)$ such that a_n was the special case $F(n, 1, 1, \dots)$, in which all the extra parameters $I_1, I_2, \dots$, are set to 1. Since the system of linear recurrence equations always seems to have constant coefficients, the generating function in all the corresponding continuous variables

$$\tilde{F}(z; x_1, x_2, \dots) := \sum_{n, I_1, I_2, \dots} F(n, I_1, I_2, \dots) z^n x_1^{I_1} x_2^{I_2} \dots$$

should be holonomic (multi-D-finite) in all its variables (because of the nonstandard boundary conditions, it is not always a rational function). It follows from the holonomic theory [9] that any coefficient with respect to $x_1, x_2, \dots$, in particular that of $x_1^1 x_2^1 \dots$, is still holonomic (D-finite, i.e. satisfies a linear differential equation with polynomial coefficients), hence $f_n := F(n, 1, 1, \dots)$ is $P-$recursive.

The method described here works for many patterns and sets of patterns, but it does not seem to work for all patterns. The authors were unable to find a suitable set of parameters (see the method, step 2) to apply this method to the pattern $adcb$.

Appendix

A maple package which confirms and illustrates many results from this paper is available and can be obtained using your favorite world wide web browser at `http://www.math.temple.edu/~noonan` or `http://www.math.temple.edu/~zeilberg`.

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