Sketch of a Proof of an Intriguing Conjecture of Karola Mészáros and Alejandro Morales Regarding the Volume of the D_n Analog of the Chan-Robbins-Yuen Polytope (Or: The Morris-Selberg Constant Term Identity Strikes Again!)

By Doron ZEILBERGER

To Dick Askey (b. June 4, 1933): from a 4^3-year-old to a 3^4-year-old, and thanks for preaching the importance of Constant Term Identities!

Disclaimer: Some of the steps below (in particular, the "change of variable" in contour-integrals) require 'rigorous' justification, that I am sure could be easily supplied by a skilled analyst.

Note added July 14, 2014: Jang Soo Kim independently found the same proof, **but** his proof is complete! See `http://arxiv.org/abs/1407.3467`.

Recall that for any rational function $f(z)$ of a variable z, $CT_z f(z)$ is the coeff. of z^0 in the formal Laurent expansion of $f(z)$ (that always exists!).

Karola Mészáros and Alejandro Morales have recently made the following intriguing conjecture.

Conjecture ([MeMora], Conj. 7.12, also presented by Morales[Mora] at Stanley@70)

$$CT_{x_n}\, CT_{x_{n-1}} \,\dots\, CT_{x_1} \prod_{i=1}^{n} x_i^{-1}(1-x_i)^{-2} \prod_{1\le i<j\le n} (x_j-x_i)^{-1}(1-x_j-x_i)^{-1} \quad = \quad 2^{n^2} \prod_{k=1}^{n} Cat(k) \quad ,$$

where $Cat(k)$ are the ubiquitous Catalan numbers $(2k)!/(k!(k+1)!)$ (that Igor Pak believes are better than primes for searching for ETI!)

The present conjecture is a D_n-analog of a conjecture made in [CRY], and proved in [Z], using the versatile Morris-Selberg Constant Term Identity ([Morr], restated in [Z]):

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n}(1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b} \prod_{1\le i<j\le n} (x_j-x_i)^{-2c} = S_n(a,b,c) \quad , \qquad (Chip)$$

where

$$S_n(a,b,c) \;:=\; \frac{1}{n!}\prod_{j=0}^{n-1} \frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)} \quad .$$

By using Cauchy's theorem, this is equivalent to

$$(\frac{1}{2\pi i})^n \int_C \prod_{i=1}^{n}(1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b-1} \prod_{1\le i<j\le n} (x_j-x_i)^{-2c} \prod_{i=1}^{n} d\,x_i \;=\; S_n(a,b,c) \quad , \qquad (Atle)$$

where *now* a,b,c can be *any* real numbers (with obvious conditions to ensure convergence) and C is *any* multi-contour in n-dimensional complex space, that is far enough from the origin.

The Mészáros-Morales conjecture is the special case $a=2$, $c=\frac{1}{2}$ of the following fact.

Fact : Let a be a positive integer, and c a positive half-integer, then

$$CT_{x_n}\, CT_{x_{n-1}}\, \dots\, CT_{x_1} \prod_{i=1}^{n} x_i^{-(a-1)}(1-x_i)^{-a} \prod_{1\le i<j\le n} (x_j-x_i)^{-2c}(1-x_j-x_i)^{-2c}$$

$$= 2^{2cn(n-1)+2(a-1)n} \cdot S_n(a\,,\, -\frac{1}{2}\,,\, c) \quad .$$

Proof: Converting the iterated constant-terms to a multi-contour-integral, we have to evaluate

$$(\frac{1}{2\pi i})^n \int_C \prod_{i=1}^{n} x_i^{-a}(1-x_i)^{-a} \prod_{1\le i<j\le n} (x_j-x_i)^{-2c}(1-x_j-x_i)^{-2c} \prod_{i=1}^{n} d\, x_i \quad .$$

Now make the *change of variables*:

$$x_i = \frac{1-z_i}{2} \quad , \quad (1\le i\le n) \quad ,$$

getting that our multi-integral equals

$$(\frac{1}{2\pi i})^n \cdot 2^{2an} \cdot 2^{2cn(n-1)} (-\frac{1}{2})^n \int_{C'} \prod_{i=1}^{n} (1-z_i)^{-a}(1+z_i)^{-a} \prod_{1\le i<j\le n} (z_i-z_j)^{-2c}(z_i+z_j)^{-2c} \prod_{i=1}^{n} d\, z_i$$

$$= (\frac{1}{2\pi i})^n \cdot (-1)^n 2^{2an+2cn(n-1)-n} \int_{C'} \prod_{i=1}^{n} (1-z_i^2)^{-a} \prod_{1\le i<j\le n} (z_i^2-z_j^2)^{-2c} \prod_{i=1}^{n} d\, z_i \quad ,$$

where C' is some other multi-contour. Now is time for *yet another* **change of variable**

$$w_i = z_i^2 \quad , \quad (1\le i\le n) \quad ,$$

and we have

$$\prod_{i=1}^{n} d\, z_i = (\frac{1}{2})^n \prod_{i=1}^{n} \frac{d\, w_i}{w_i^{1/2}} \quad ,$$

getting that our multi-integral is

$$(-1)^{n+n(n-1)c} (\frac{1}{2\pi i})^n\, 2^{2an+2cn(n-1)-2n} \int_{C''} \prod_{i=1}^{n} (1-w_i)^{-a} \prod_{i=1}^{n} w_i^{-1/2} \prod_{1\le i<j\le n} (w_j-w_i)^{-2c} \prod_{i=1}^{n} d\, w_i \quad ,$$

for yet another multi-contour, C'', and thanks to $(Atle)$ this is

$$2^{2cn(n-1)+2(a-1)n} \cdot S_n(a\,,\, -\frac{1}{2}\,,\, c) \quad ,$$

times $(-1)^{n+n(n-1)c}$, and I am sure that this annoying extra sign could be explained away. $\Box$

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Slides are available here: `http://math.mit.edu/stanley70/Site/Slides/Morales.pdf`

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`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cry.html`.

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