Searching for Disjoint Covering Systems with Precisely One Repeated Modulus

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Introduction

A set of arithmetical sequences

$$a_1 \pmod{m_1} \quad , \quad a_2 \pmod{m_2} \quad , \quad \dots \quad , \quad a_k \pmod{m_k} \qquad ,$$

with

$$m_1 \leq m_2 \leq \dots \leq m_k \qquad ,$$

is called a *disjoint covering system* (alias *exact covering system*) if every positive integer belongs to **exactly** one of the sequences.

Mirski, Newman, Davenport and Rado (See [3]) famously proved that the moduli can't all be distinct. In fact the two largest moduli must be equal, i.e.

$$m_{k-1} = m_k \quad .$$

This raises the natural question:

How close can you get to getting distinct moduli?

Of course, for any positive integer r, the system

$$0 \pmod{2} \quad , \quad 1 \pmod{4} \quad , \quad 3 \pmod{8} \quad , \quad \dots \quad ,$$

$$2^{r-2}-1 \,(\mathrm{mod} 2^{r-1}) \quad , \quad 2^{r-1}-1 \,(\mathrm{mod} 2^{r}) \quad , \quad 2^{r}-1 \,(\mathrm{mod} 2^{r}) \quad ,$$

has the property that all the moduli are distinct except for the largest modulus, 2^r, that only appears twice. So there are *infinitely* many systems where the moduli are all distinct except the largest moduli repeating twice. We label these as *trivial.*

More generally, for any given positive integer r, once you have one system with all distinct moduli, except for the largest, that is repeated, r times, one can manufacture infinitely many examples, by the "add-2" process. Multiply all the moduli by 2 and prefix 2 at the beginning.

But if you insist that the smallest modulus is at least three, then, for any given r, there are (conjecturally, but most probably) only finitely many such systems. This raises the natural question of finding all of them for small r.

This was done by Berger, Felzenbaum, and Fraenkel([2]), for $3 \leq r \leq 9$, and extended by Zeleke and Simpson for $10 \leq r \leq 12$ ([4]). In this article we continue the list all the way to $r = 32$.

We must confess that, while we have proved that all the systems that we have found are indeed correct, we did not bother to prove that all our lists are complete. What we do know, *for sure*, is that we found all such systems where the largest modulus is ≤ 600. Nevertheless we strongly believe, on heuristic and experimental grounds, that there aren't any new ones.

The results below were found using the Maple package `BFF.txt`, available, free of charge, from the following url:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/BFF.txt` .

The front of the present article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ecs.html` ,

contains links to sample input and output files, that the readers are welcome to extend, if they have computer time to spare.

The output: the list of all Exact Covering Systems whose smallest modulus is at least 3 and whose moduli are all distinct except the largest one that is repeated at most 32 times

We exclude from the list the trivial systems consisting of only one modulus.

2 Repeats

None

3 Repeats

None

4 Repeats

There is 1 system here it is:

3 6 (4 times)

5 Repeats

None

6 Repeats

There are 3 systems here they are:

3 9 (6 times)

4 8 (6 times)

3 6 12 (6 times)

7 Repeats

There are 2 systems here they are:

4 6 12 (7 times)

3 6 9 18 (7 times)

8 Repeats

There are 2 systems here they are:

3 12 (8 times)

5 10 (8 times)

9 Repeats

There are 4 systems here they are:

4 12 (9 times)

3 6 18 (9 times)

4 6 8 12 24 (9 times)

3 6 9 12 18 36 (9 times)

10 Repeats

There are 5 systems here they are:

3 15 (10 times)

6 12 (10 times)

3 9 18 (10 times)

4 8 16 (10 times)

3 6 12 24 (10 times)

11 Repeats

There are 2 systems here they are:

4 6 8 24 (11 times)

3 6 9 12 36 (11 times)

12 Repeats

There are 7 systems here they are:

3 18 (12 times)

4 16 (12 times)

5 15 (12 times)

7 14 (12 times)

3 6 24 (12 times)

4 6 12 24 (12 times)

3 6 9 18 36 (12 times)

13 Repeats

There are 7 systems here they are:

4 10 20 (13 times)

6 9 18 (13 times)

3 6 15 30 (13 times)

4 8 12 24 (13 times)

3 6 12 18 36 (13 times)

4 6 8 12 16 24 48 (13 times)

3 6 9 12 18 24 36 72 (13 times)

14 Repeats

There are 6 systems here they are:

3 21 (14 times)

8 16 (14 times)

3 12 24 (14 times)

4 6 24 (14 times)

5 10 20 (14 times)

3 6 9 36 (14 times)

15 Repeats

There are 10 systems here they are:

4 20 (15 times)

6 18 (15 times)

3 6 30 (15 times)

3 9 27 (15 times)

4 8 24 (15 times)

3 6 12 36 (15 times)

6 8 12 24 (15 times)

3 9 12 18 36 (15 times)

4 6 8 12 16 48 (15 times)

3 6 9 12 18 24 72 (15 times)

16 Repeats

There are 9 systems here they are:

3 24 (16 times)

5 20 (16 times)

9 18 (16 times)

4 12 24 (16 times)

3 6 18 36 (16 times)

4 6 12 18 36 (16 times)

3 6 9 18 27 54 (16 times)

4 6 8 12 24 48 (16 times)

3 6 9 12 18 36 72 (16 times)

17 Repeats

There are 4 systems here they are:

6 8 24 (17 times)

3 9 12 36 (17 times)

4 6 8 16 24 48 (17 times)

3 6 9 12 24 36 72 (17 times)

18 Repeats

There are 16 systems here they are:

3 27 (18 times)

4 24 (18 times)

7 21 (18 times)

10 20 (18 times)

3 6 36 (18 times)

3 15 30 (18 times)

6 12 24 (18 times)

3 9 18 36 (18 times)

4 6 12 36 (18 times)

4 8 16 32 (18 times)

3 6 9 18 54 (18 times)

3 6 12 24 48 (18 times)

4 6 8 12 48 (18 times)

3 6 9 12 18 72 (18 times)

4 6 8 12 18 24 36 72 (18 times)

3 6 9 12 18 27 36 54 108 (18 times)

19 Repeats

There are 13 systems here they are:

4 14 28 (19 times)

8 12 24 (19 times)

3 6 21 42 (19 times)

3 12 18 36 (19 times)

4 6 18 36 (19 times)

5 10 15 30 (19 times)

3 6 9 27 54 (19 times)

4 6 8 16 48 (19 times)

4 8 10 20 40 (19 times)

3 6 9 12 24 72 (19 times)

3 6 12 15 30 60 (19 times)

4 6 12 16 24 48 (19 times)

3 6 9 18 24 36 72 (19 times)

20 Repeats

There are 12 systems here they are:

3 30 (20 times)

5 25 (20 times)

6 24 (20 times)

11 22 (20 times)

3 9 36 (20 times)

4 8 32 (20 times)

3 6 12 48 (20 times)

6 10 15 30 (20 times)

4 6 8 24 48 (20 times)

3 6 9 12 36 72 (20 times)

4 6 8 12 18 24 72 (20 times)

3 6 9 12 18 27 36 108 (20 times)

21 Repeats

There are 18 systems here they are:

4 28 (21 times)

8 24 (21 times)

3 6 42 (21 times)

3 12 36 (21 times)

4 6 36 (21 times)

5 10 30 (21 times)

3 6 9 54 (21 times)

4 8 10 40 (21 times)

3 6 12 15 60 (21 times)

4 6 12 16 48 (21 times)

6 9 12 18 36 (21 times)

3 6 9 18 24 72 (21 times)

4 8 12 16 24 48 (21 times)

3 6 12 18 24 36 72 (21 times)

4 6 8 12 18 36 72 (21 times)

3 6 9 12 18 27 54 108 (21 times)

4 6 8 12 16 24 32 48 96 (21 times)

3 6 9 12 18 24 36 48 72 144 (21 times)

22 Repeats

There are 19 systems here they are:

3 33 (22 times)

12 24 (22 times)

3 18 36 (22 times)

4 16 32 (22 times)

5 15 30 (22 times)

6 10 30 (22 times)

7 14 28 (22 times)

3 6 24 48 (22 times)

3 9 15 45 (22 times)

4 6 8 48 (22 times)

4 12 18 36 (22 times)

3 6 9 12 72 (22 times)

3 6 18 27 54 (22 times)

4 6 12 24 48 (22 times)

3 6 9 18 36 72 (22 times)

4 6 8 12 24 36 72 (22 times)

3 6 9 12 18 36 54 108 (22 times)

4 6 8 12 16 18 24 36 48 72 144 (22 times)

3 6 9 12 18 24 27 36 54 72 108 216 (22 times)

23 Repeats

There are 12 systems here they are:

6 15 30 (23 times)

4 8 20 40 (23 times)

6 9 12 36 (23 times)

3 6 12 30 60 (23 times)

4 6 16 24 48 (23 times)

4 8 12 16 48 (23 times)

3 6 9 24 36 72 (23 times)

3 6 12 18 24 72 (23 times)

4 6 8 12 18 72 (23 times)

3 6 9 12 18 27 108 (23 times)

4 6 8 12 16 24 32 96 (23 times)

3 6 9 12 18 24 36 48 144 (23 times)

24 Repeats

There are 24 systems here they are:

3 36 (24 times)

4 32 (24 times)

5 30 (24 times)

7 28 (24 times)

9 27 (24 times)

13 26 (24 times)

3 6 48 (24 times)

4 12 36 (24 times)

3 6 18 54 (24 times)

4 6 12 48 (24 times)

4 10 20 40 (24 times)

6 9 18 36 (24 times)

3 6 9 18 72 (24 times)

3 6 15 30 60 (24 times)

4 8 12 24 48 (24 times)

3 6 12 18 36 72 (24 times)

4 6 8 12 24 72 (24 times)

3 6 9 12 18 36 108 (24 times)

4 6 8 18 24 36 72 (24 times)

3 6 9 12 27 36 54 108 (24 times)

4 6 8 12 16 24 48 96 (24 times)

3 6 9 12 18 24 36 72 144 (24 times)

4 6 8 12 16 18 24 36 48 144 (24 times)

3 6 9 12 18 24 27 36 54 72 216 (24 times)

25 Repeats

There are 23 systems here they are:

6 30 (25 times)

3 9 45 (25 times)

4 8 40 (25 times)

4 18 36 (25 times)

10 15 30 (25 times)

3 6 12 60 (25 times)

3 6 27 54 (25 times)

4 6 16 48 (25 times)

6 12 18 36 (25 times)

3 6 9 24 72 (25 times)

3 9 18 27 54 (25 times)

4 8 16 24 48 (25 times)

3 6 12 24 36 72 (25 times)

4 6 8 12 36 72 (25 times)

4 6 12 20 30 60 (25 times)

6 8 12 16 24 48 (25 times)

3 6 9 12 18 54 108 (25 times)

3 6 9 18 30 45 90 (25 times)

3 9 12 18 24 36 72 (25 times)

4 6 8 12 16 32 48 96 (25 times)

3 6 9 12 18 24 48 72 144 (25 times)

4 6 8 12 16 18 24 36 72 144 (25 times)

3 6 9 12 18 24 27 36 54 108 216 (25 times)

26 Repeats

There are 19 systems here they are:

3 39 (26 times)

14 28 (26 times)

3 21 42 (26 times)

4 10 40 (26 times)

6 9 36 (26 times)

8 16 32 (26 times)

3 6 15 60 (26 times)

3 12 24 48 (26 times)

4 6 24 48 (26 times)

4 8 12 48 (26 times)

5 10 20 40 (26 times)

3 6 9 36 72 (26 times)

3 6 12 18 72 (26 times)

4 6 8 18 24 72 (26 times)

3 6 9 12 27 36 108 (26 times)

4 6 8 12 16 24 96 (26 times)

3 6 9 12 18 24 36 144 (26 times)

4 6 8 12 16 18 24 48 72 144 (26 times)

3 6 9 12 18 24 27 36 72 108 216 (26 times)

27 Repeats

There are 27 systems here they are:

4 36 (27 times)

10 30 (27 times)

3 6 54 (27 times)

3 15 45 (27 times)

6 12 36 (27 times)

3 9 18 54 (27 times)

4 8 16 48 (27 times)

9 12 18 36 (27 times)

3 6 12 24 72 (27 times)

4 6 8 12 72 (27 times)

4 6 12 20 60 (27 times)

4 12 16 24 48 (27 times)

6 8 12 16 48 (27 times)

3 6 9 12 18 108 (27 times)

3 6 9 18 30 90 (27 times)

3 6 18 24 36 72 (27 times)

3 9 12 18 24 72 (27 times)

4 6 8 18 36 72 (27 times)

3 6 9 12 27 54 108 (27 times)

4 6 8 12 16 32 96 (27 times)

4 6 12 18 24 36 72 (27 times)

3 6 9 12 18 24 48 144 (27 times)

3 6 9 18 27 36 54 108 (27 times)

4 6 8 12 24 32 48 96 (27 times)

3 6 9 12 18 36 48 72 144 (27 times)

4 6 8 12 16 18 24 36 144 (27 times)

3 6 9 12 18 24 27 36 54 216 (27 times)

28 Repeats

There are 26 systems here they are:

3 42 (28 times)

5 35 (28 times)

8 32 (28 times)

15 30 (28 times)

3 12 48 (28 times)

4 6 48 (28 times)

4 20 40 (28 times)

5 10 40 (28 times)

6 18 36 (28 times)

3 6 9 72 (28 times)

3 6 30 60 (28 times)

3 9 27 54 (28 times)

4 8 24 48 (28 times)

3 6 12 36 72 (28 times)

4 6 12 30 60 (28 times)

6 8 12 24 48 (28 times)

3 6 9 18 45 90 (28 times)

3 9 12 18 36 72 (28 times)

4 6 8 24 36 72 (28 times)

3 6 9 12 36 54 108 (28 times)

4 6 8 12 16 48 96 (28 times)

3 6 9 12 18 24 72 144 (28 times)

4 6 8 12 16 18 24 48 144 (28 times)

3 6 9 12 18 24 27 36 72 216 (28 times)

4 6 8 12 16 18 36 48 72 144 (28 times)

3 6 9 12 18 24 27 54 72 108 216 (28 times)

29 Repeats

There are 21 systems here they are:

9 12 36 (29 times)

4 12 16 48 (29 times)

8 10 20 40 (29 times)

3 6 18 24 72 (29 times)

3 12 15 30 60 (29 times)

4 6 8 18 72 (29 times)

4 8 14 28 56 (29 times)

6 8 16 24 48 (29 times)

3 6 9 12 27 108 (29 times)

3 6 12 21 42 84 (29 times)

3 9 12 24 36 72 (29 times)

4 6 12 18 24 72 (29 times)

4 10 12 20 30 60 (29 times)

3 6 9 18 27 36 108 (29 times)

3 6 15 18 30 45 90 (29 times)

4 6 8 12 24 32 96 (29 times)

3 6 9 12 18 36 48 144 (29 times)

4 6 8 16 24 32 48 96 (29 times)

3 6 9 12 24 36 48 72 144 (29 times)

4 6 8 12 16 18 24 72 144 (29 times)

3 6 9 12 18 24 27 36 108 216 (29 times)

30 Repeats

There are 39 systems here they are:

3 45 (30 times)

4 40 (30 times)

6 36 (30 times)

7 35 (30 times)

11 33 (30 times)

16 32 (30 times)

3 6 60 (30 times)

3 9 54 (30 times)

3 24 48 (30 times)

4 8 48 (30 times)

5 20 40 (30 times)

9 18 36 (30 times)

3 6 12 72 (30 times)

4 6 12 60 (30 times)

4 12 24 48 (30 times)

6 8 12 48 (30 times)

6 14 21 42 (30 times)

3 6 9 18 90 (30 times)

3 6 18 36 72 (30 times)

3 9 12 18 72 (30 times)

4 6 8 24 72 (30 times)

4 6 20 30 60 (30 times)

3 6 9 12 36 108 (30 times)

3 6 9 30 45 90 (30 times)

4 6 8 12 16 96 (30 times)

4 6 12 18 36 72 (30 times)

3 6 9 12 18 24 144 (30 times)

3 6 9 18 27 54 108 (30 times)

4 6 8 12 24 48 96 (30 times)

4 8 12 18 24 36 72 (30 times)

6 10 12 15 20 30 60 (30 times)

3 6 9 12 18 36 72 144 (30 times)

3 6 12 18 27 36 54 108 (30 times)

4 6 8 12 16 18 36 48 144 (30 times)

3 6 9 12 18 24 27 54 72 216 (30 times)

4 6 8 12 16 24 36 48 72 144 (30 times)

3 6 9 12 18 24 36 54 72 108 216 (30 times)

4 6 8 12 16 18 24 32 36 48 72 96 144 288 (30 times)

3 6 9 12 18 24 27 36 48 54 72 108 144 216 432 (30 times)

31 Repeats

There are 35 systems here they are:

4 22 44 (31 times)

8 10 40 (31 times)

12 18 36 (31 times)

3 6 33 66 (31 times)

3 12 15 60 (31 times)

3 18 27 54 (31 times)

4 8 14 56 (31 times)

4 16 24 48 (31 times)

6 8 16 48 (31 times)

7 14 21 42 (31 times)

3 6 12 21 84 (31 times)

3 6 24 36 72 (31 times)

3 9 12 24 72 (31 times)

4 6 8 36 72 (31 times)

4 10 12 20 60 (31 times)

6 12 16 24 48 (31 times)

3 6 9 12 54 108 (31 times)

3 6 15 18 30 90 (31 times)

3 9 18 24 36 72 (31 times)

4 6 12 24 36 72 (31 times)

3 6 9 18 36 54 108 (31 times)

4 6 8 12 32 48 96 (31 times)

4 6 8 16 24 32 96 (31 times)

4 8 10 16 20 40 80 (31 times)

3 6 9 12 18 48 72 144 (31 times)

3 6 9 12 24 36 48 144 (31 times)

3 6 12 15 24 30 60 120 (31 times)

4 6 8 12 16 18 24 144 (31 times)

3 6 9 12 18 24 27 36 216 (31 times)

4 6 8 12 16 18 36 72 144 (31 times)

4 6 8 12 24 30 40 60 120 (31 times)

3 6 9 12 18 24 27 54 108 216 (31 times)

3 6 9 12 18 36 45 60 90 180 (31 times)

4 6 8 12 18 24 36 48 72 144 (31 times)

3 6 9 12 18 27 36 54 72 108 216 (31 times)

32 Repeats

There are 29 systems here they are:

3 48 (32 times)

5 40 (32 times)

9 36 (32 times)

17 34 (32 times)

4 12 48 (32 times)

6 14 42 (32 times)

3 6 18 72 (32 times)

3 9 21 63 (32 times)

4 6 20 60 (32 times)

6 8 24 48 (32 times)

3 6 9 30 90 (32 times)

3 9 12 36 72 (32 times)

4 6 12 18 72 (32 times)

4 10 12 30 60 (32 times)

3 6 9 18 27 108 (32 times)

3 6 15 18 45 90 (32 times)

4 6 8 12 24 96 (32 times)

4 8 12 18 24 72 (32 times)

6 10 12 15 20 60 (32 times)

3 6 9 12 18 36 144 (32 times)

3 6 12 18 27 36 108 (32 times)

4 6 8 16 24 48 96 (32 times)

3 6 9 12 24 36 72 144 (32 times)

4 6 8 12 16 18 48 72 144 (32 times)

4 6 8 12 16 24 36 48 144 (32 times)

3 6 9 12 18 24 27 72 108 216 (32 times)

3 6 9 12 18 24 36 54 72 216 (32 times)

4 6 8 12 16 18 24 32 36 48 72 96 288 (32 times)

3 6 9 12 18 24 27 36 48 54 72 108 144 432 (32 times)

Final Comments

1. For a small number of repeats, the present computations and those done previously, confirm the proved and conjectured structures of such systems with a single repeated modulus. It is an open problem to characterize the structure for an arbitrary number of repeats.

2. Another direction of investigation would be to compute the number of repeats when we have exactly two distinct repeated moduli. A lower bound on the number of repeats for any number of repeated moduli is given in [1]. This direction may be conducive for solving **1**.

3. This article could have been written in many fewer pages (and had we published it in the obsolete medium of print-journal, it would have been a good idea, in order to save trees), by presenting the data concisely in a table, where the values of r appear in the leftmost column. Thus, for example, the table entry for $r = 14$ would be:

14(6): 3, **21**; 8, **16**; 3,12, **24**; 4,6, **24**; 5,10, **20**; 3,6, 9, **36** .

(The parenthesized number next to r is the number of systems. The number in boldface is the modulus that's repeated r times.)

But for the sake of *readability* and *clarity*, and with the hope that no one of our readers will be stupid enough to print out this paper, we decided to keep it in the present form.

References

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3. P. Erdős, On a problem concerning congruence systems. (Hungarian, with English summary) Mat. Lapok **3**(1952), 122-128.

4. M. Zeleke and J. Simpson, *On Disjoint Covering Systems with precisely one repeated modulus*, Advances in Applied Mathematics **23**(1999), 322-332.

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