DETERMINANTS THROUGH THE LOOKING GLASS

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Submitted: December 23, 2000.

Dedicated to Dominique Foata on his 65^{th}-birthday

Abstract. Using a recurrence derived from Dodgson's Condensation Method, we provide numerous explicit evaluations of determinants. They were all conjectured, and then rigorously proved, by computer-assisted methods, that should be amenable to full automation. We also mention a first step towards that goal, our Maple package, `DODGSON`, that automates the special case of Hankel and Toeplitz hypergeometric determinants.

AMS Subject Classification: Primary 05A, 15A

This article is motivated by the computation in [1] that was inspired by the short proof [6] of MacMahon's determinant evaluation [4], using a determinantal identity of Charles Dodgson [2]. Many special cases of the sampled determinants given below were independently discovered by M. Petkovšek [5]. For an excellent and detailed survey of existing methods of proofs of determinant identities, see [3].

For any n by n matrix A, let $A_r(i,j)$ denote the r by r minor consisting of r contiguous rows and columns of A, starting with row i and column j. In particular, $A_n(1,1) = \det A$. Then, according to Dodgson [2],

$$(Lewis) \qquad A_n(1,1)A_{n-2}(2,2) = A_{n-1}(1,1)A_{n-1}(2,2) - A_{n-1}(2,1)A_{n-1}(1,2).$$

For many cases, $A_n(i,j)$ turn out, conjectured at first, to have an explicit expression, involving single and double products. Whenever this is the case the proof of the conjectured evaluation is completely routine, by induction on n, by checking that *(Lewis)* is satisfied by that conjectured expression, and by checking the trivial initial conditions for $n=0$ and $n=1$. Finally, to get an explicit expression for the original determinant, all one has to do is plug in $i=1$ and $j=1$.

A more interesting case happens when $A_n(i,j)$ does not seem to have an explicit expression, yet $A_n(1,1)$ does. We believe that in many cases, *(Lewis)* should still be useful, by extending the *ansatz* to a larger class, that for us humans looks messy, but that computers won't mind. Then plugging in $i=1$ and $j=1$ in that 'messy expression' (which may well be a recurrence satisfied by it) could still be simplified to something 'nice.'

Be that as it may, in the former case things could be made completely automatic. But this programming chore is too daunting. Hence so far we only accomplished a semi-automated implementation of the special case of Hankel and Toeplitz determinants of hypergeometric type. The reader is invited to check out our Maple package `DODGSON`, that has on-line help.

Examples of Computer-assisted Explicit Evaluations:

(1)
$$\det\left[\frac{1}{i!}\binom{x+ai+j}{y+ai-j}_{0\le i,j\le n}\right]$$

$$= a^{\binom{n+1}{2}}\prod_{i=0}^{n}\frac{(x+ai)!}{(y+ai)!(x-y+2i)!}\prod_{j=1}^{i}\left[x+y+(i+j-1)a+1\right].$$

(2)
$$\det\left[\frac{1}{i!}\binom{x+ai+j}{y+ai-j}^{-1}_{0\le i,j\le n}\right]$$

$$= (-a)^{\binom{n+1}{2}}\prod_{i=0}^{n}\frac{(x-y+2i)!(y+ai-n)!}{(x+ai+n)!}\prod_{j=1}^{i}\left[x+y+(i+j-1)a+1\right].$$

(3)
$$\det\left[\binom{2x+2ai+2j}{x+ai+j}_{0\le i,j\le n}\right]$$

$$= 2^n n!^{n+1}a^{\binom{n+1}{2}}\prod_{i=0}^{n}\binom{n+i}{i}\binom{2i}{i}^{-1}\binom{2x+2ai}{x+ai}\prod_{j=1}^{n}\frac{1}{[x+ai+j]}.$$

(4)
$$\det\left[i!\binom{x+ai+cj}{j}\binom{y+bi-j}{n-j}_{0\le i,j\le n}\right]$$

$$= \prod_{j=1}^{n}\prod_{i=1}^{j}\left[(y-j+1)a-(x+cj-i+1)b\right],$$

$$\det\left[i!\binom{x+ai+j}{j}\binom{y+bi+cj}{n-j}\right]_{0\le i,j\le n} \tag{5}$$

$$=\prod_{i=1}^{n}\prod_{j=1}^{i}\left[(y+(n-j)c-i+1)a-(x+n-i+1)b\right],$$

$$\det\left[\frac{1}{i!}\binom{x+ai+cj}{j}\binom{y+bi+j}{j}^{-1}\right]_{0\le i,j\le n} \tag{6}$$

$$=\frac{\prod_{j=0}^{n}(y+bj)!}{\prod_{j=0}^{n}(y+bj+n)!}\prod_{j=1}^{n}\prod_{i=1}^{j}\left[(y+j)a-(x+cj-i+1)b\right],$$

$$\det\left[\frac{x+ai+bj}{y+ai+bj}\right]_{0\le i,j\le n} \tag{7}$$

$$=(ab)^{\binom{n+1}{2}}(x-y)^n\left[x+ny+\binom{n+1}{2}(a+b)\right]\frac{\prod_{i=0}^{n}i!^2}{\prod_{i,j=0}^{n}\left[y+ai+bj\right]}.$$

$$\det\left[(x+ai+bj)^k\right]_{0\le i,j\le n}=\begin{cases} n!^{n+1}(-ab)^{\binom{n+1}{2}}, & \text{if } k=n\\ 0, & \text{if } 0\le k<n.\end{cases} \tag{8}$$

$$\det\left[(x+ai+bj)^n(y+ai-bj)^n\right]_{0\le i,j\le n} \tag{9}$$

$$=n!^{n+1}(ab)^{\binom{n+1}{2}}\prod_{j=1}^{n}\prod_{i=1}^{j}\left[x+y+(i+j-1)a\right]\left[x-y+(i+j-1)b\right].$$

$$\det\left[\frac{(x+ai+j)(y+bi+j)}{x+(a+b)i+j}\right]_{0\le i,j\le n} \tag{10}$$

$$= b^n(a+b)^{\binom{n}{2}}\left[y+nx+\binom{n+1}{2}(a+b+1)\right]\prod_{i=1}^{n} i!^2[y-x-ia]\prod_{j=0}^{n}\frac{1}{[x+(a+b)i+j]}.$$

$$\det\left[\frac{1}{(x+bi+j)(y-bi+j)}\right]_{0\le i,j\le n} \tag{11}$$

$$= (-b)^{\binom{n+1}{2}}\prod_{i,j=0}^{n}\frac{1}{(x+bi+j)(y-bi+j)}\prod_{j=1}^{n} j!^2\prod_{i=1}^{j}[x-y+(j+i-1)b][x+y+j+i-1].$$

$$\det\left[\frac{1}{(x+ai+j)(a+x+ai+j)}\right]_{0\le i,j\le n} \tag{12}$$

$$= \frac{a^{\binom{n+1}{2}}}{(n+1)!}\prod_{i=0}^{n+1} i!^2\prod_{j=0}^{n}\frac{1}{(x+ai+j)}.$$

$$\det\left[\frac{(x+ai+j-1)!}{(x+r+ai+j)!}\right]_{0\le i,j\le n} \tag{13}$$

$$= a^{\binom{n+1}{2}}\prod_{i=0}^{n} i!^2\binom{r+i}{i}\frac{(x+ai-1)!}{(x+r+ai+n)!}.$$

Remark: Notice in particular that Eqn. (13) reveals the Hilbert matrix when $r=0$ and $a=1$.

$$\det\left[j!\binom{x+ai+cj}{j}\binom{y-ai+cj}{j}\right]_{0\le i,j\le n} \tag{14}$$

$$= (-a)^{\binom{n+1}{2}}\prod_{j=1}^{n}\prod_{i=1}^{j}[x-y+(i+j-1)a].$$

We now demonstrate how such identities could be discovered. Let us take the following example:

$$(15)\qquad \det\left[\frac{1}{(r+i+j)(1+r+i+j)}\right]_{0\le i,j\le n} = \frac{1}{(n+1)!}\prod_{i=0}^{n+1} i!^2 \prod_{j=0}^{n}\frac{1}{(r+i+j)}.$$

Denoting the left-hand side of (15) by $f_n(r)$, we form the ratio $h_n(r) := \frac{f_n(r+1)}{f_n(r)}$ and input empirical data (n fixed, r varying) in the MAPLE package `gfun`, which in turn suggests the recurrence

$$-(r+n+1)(r+n+2)h_n(r) + r(r+2n+3)h_n(r+1) = 0$$

for $h_n(r)$. This, combined with the definition of $h_n(r)$ implies that

$$f_n(r) = a_n \frac{(r+n)!!(r+n-1)!!}{(r-2)!!(r+2n+1)!!}$$

for some constant a_n, depending (possibly) on n. At this stage, we invoke the recurrence relation (*Lewis*), resulting from Dodgson's rule, on $f_n(r)$. Consequently, we obtain

$$\frac{a_n a_{n-2}}{a_{n-1}^2} = n(n+1).$$

We then conclude that $a_n = n!!(n+1)!!$ and the construction of the identity (15) is completed.

Our Maple package `DODGSON`, combines some of these intermediate steps for the special cases of Hankel and Toeplitz determinants of hypergeometric type. `DODGSON` is available at `http://www.math.temple.edu/~ zeilberg/tokhniot/DODGSON.`

Further Notes:

Let $P(i,j,x)$ be polynomials in x, and assume also $P(i+1,j,x) = P(i,j,x+c)$ for some constant c. Then, we have

Fact 1: If $deg(P(i,j,x) < n$,

$$\det[P(i,j,x)]_{0\le i,j\le n} = 0.$$

Proof: Follows from a rank argument on the first row of the matrix and the linearity assumption, above. □

Fact 2: If in addition, P is of degree n and $P(i, j, x) = g(i + j + x)$, then

$$\det[P(i,j,x)]_{0\le i,j\le n} \equiv \textit{constant.}$$

Proof: By embedding the given matrix $M_n(x) := [P(i,j,x)]_{0\le i,j\le n}$ into $M_{n,0}(x) := [P(i,j,x)]_{0\le i,j\le n+1}$ and applying Dodgson's rule, we gather that the determinant $f_n(x) := \det(M_n(x))$ of the original matrix satisfies

$$f_n(x)f_n(x+2) = f_n(x+1)^2, \tag{16}$$

since the determinant of the new matrix $M_{n,0}(x)$ vanishes by Fact 1, above. But $f_n(x)$ is a polynomial, thus for Eqn. (16) to hold $f_n(x)$ must be a constant. $\square$

Postscript:

Christian Krattenthaler pointed out that most of our determinants are special cases of known determinants mentioned in [3]. More interestingly, with the exception of our identities (8)-(10), they can be derived from his amazingly general and versatile lemma [3, Lemma 5.] He also recommended the Maple package "Guess" by Béraud and Gauthier for more efficient guessing.

Nevertheless, the identities we presented above are all beautiful, and once our complete automation will be achieved, it would be much easier to prove them from scratch than to find how they can be derived from Krattenthaler's Lemma. Also, in defense of Dodgson, we are almost sure, (and will be glad to try it for a fee of $5000), that Krattenthaler's Lemma is Dodgeable, and the humanly-daunting task of manipulating double products should also be capable of automation.

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