The Dayenu Boolean Function Is Almost Always True!

Doron ZEILBERGER

Dedicated to Hemi and Yael Nae

[Recall, from Logic 101, that for any two statements, X, Y, $X \vee Y$ means that "either X or Y or both happened", $X \wedge Y$ means that **both** X and Y happened, while $\bar{X}$ means that X did **not** happen. For typographical clarity, $X \wedge Y$ is often written XY.]

Two nights ago, my wife Jane and I were fortunate to be guests in a wonderful Passover *seder* at the house of Hemi[1] and Yael Nae. Soon enough we came to the number-one hit song, *Dayenu*, praising God for doing 15 amazing miracles, let's call them $x_1, \dots, x_{15}$, where x_1 stands for "Took us out of Egypt", and x_{15} stands for "Built us our Temple". The exact nature of the other 13 miracles (and for that matter the above two) are irrelevant to this *mathematical* paper, but can be easily looked-up in any *Haggadah* (and of course, nowadays, on the internet).

It so happened that what God actually did can be described by the Boolean function (with $n = 15$)

$$G_n(x_1, \dots, x_n) = x_1 \wedge \dots \wedge x_n = \bigwedge_{i=1}^{n} x_i \quad ,$$

whose only *truth* vector is the all-true vector T^n, i.e., assuming, that the probability of any one miracle occurring is p (and that they are independent events), has the tiny probability of p^n.

But the author of *Dayenu* asserts that God was an over-achiever, and we, the children of Israel, should have been content if the following Boolean function would have been *satisfied*.

$$D_n(x_1, \dots, x_n) = \bigvee_{i=1}^{n-1} x_i \, \overline{x_{i+1}} \quad .$$

The following question *immediately* came to my mind: How many (and which) truth-vectors are satisfied by the *Dayenu* function, and what is the probability that God, deciding randomly which miracles to perform and which not to perform, would have satisfied the *minimum* requirement demanded by the anonymous author of *Dayenu*?

Thanks to De Morgan we have

$$\overline{D_n(x_1, \dots, x_n)} = \bigwedge_{i=1}^{n-1} (\, \overline{x_i} \vee x_{i+1} \,) \quad .$$

[1] Hemi and I were dorm-mates at the Weizmann Institute, way back in the early seventies. His son is also called Doron, so we call him 'little Doron', while I am 'big Doron'. 'Little Doron' is no longer so little (he is 25-years-old), but Hemi commented that he knows me ('big Doron') much longer than he knows 'little Doron'.

We are now ready for

The Dayenu Theorem. The full disjunctive normal form of the negation of the Dayenu Boolean function is given by

$$\overline{D_n(x_1,\dots,x_n)} = \bigvee_{i=1}^{n+1} \left(\bigwedge_{j=1}^{n-i+1} \overline{x_j} \bigwedge_{j=n-i+2}^{n} x_j \right) \quad ,$$

or, more concretely:

$$\overline{D_n(x_1,\dots,x_n)} = \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-1}}\,\overline{x_n} \vee \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-1}}\,x_n \vee \dots \vee \overline{x_1}\,x_2\dots x_{n-1}\,x_n \vee x_1\,x_2\cdots x_n \quad .$$

Proof: By induction on n. It is true when $n = 2$ (check!). Assume that it is true when n is replaced by $n-1$. Note that

$$\overline{D_n(x_1,\dots,x_n)} = \overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge (\overline{x_{n-1}} \vee x_n)$$

$$= \overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge \overline{x_{n-1}} \vee \overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge x_n \quad .$$

By the inductive hypothesis:

$$\overline{D_{n-1}(x_1,\dots,x_{n-1})} = \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}} \vee \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1} \vee \dots \vee \overline{x_1}\,x_2\dots x_{n-2}\,x_{n-1} \vee x_1\,x_2\cdots x_{n-1} \quad .$$

Regarding $\overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge \overline{x_{n-1}}$ we have

$$\overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge \overline{x_{n-1}}$$

$$= \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,\overline{x_{n-1}} \vee \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1}\,\overline{x_{n-1}} \vee \dots \vee \overline{x_1}\,x_2\dots x_{n-2}\,x_{n-1}\,\overline{x_{n-1}} \vee x_1\,x_2\cdots x_{n-1}\,\overline{x_{n-1}} \quad .$$

Since $x_{n-1}\,\overline{x_{n-1}}$ is FALSE, all the above terms, except the first, vanish, and since $\overline{x_{n-1}}\,\overline{x_{n-1}} = \overline{x_{n-1}}$, we have

$$\overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge \overline{x_{n-1}} = \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}} \quad .$$

Regarding $\overline{D_{n-1}(x_1,\dots,x_{n-1})} \wedge x_n$ we have

$$\overline{D_{n-1}(x_1,\dots,x_{n-1})}\,x_n =$$

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \vee \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1}\,x_n \vee \dots \vee \overline{x_1}\,x_2\dots x_{n-2}\,x_{n-1}\,x_n \vee x_1\,x_2\cdots x_{n-1}\,x_n \quad .$$

Combining, we have

$$\overline{D_n(x_1,\dots,x_n)} =$$

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}$$

$$\vee\, \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \vee \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1}\,x_n \vee \dots \vee \overline{x_1}\,x_2\dots x_{n-2}\,x_{n-1}\,x_n \vee x_1\,x_2\cdots x_{n-1}\,x_n \quad .$$

Since

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}} = \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,\overline{x_n} \lor \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \quad ,$$

we get

$$\overline{D_n(x_1,\dots,x_n)} =$$

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,\overline{x_n} \lor \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \lor$$

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \lor \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1}\,x_n \lor \dots \lor \overline{x_1}\,x_2\dots x_{n-2}\,x_{n-1}\,x_n \lor x_1\,x_2\cdots x_{n-1}\,x_n \quad .$$

Since the second and third term above are the same (and $X \lor X = X$), we finally get

$$\overline{D_n(x_1,\dots,x_n)} =$$

$$\overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,\overline{x_n} \lor \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,\overline{x_{n-1}}\,x_n \lor \overline{x_1}\,\overline{x_2}\cdots\overline{x_{n-2}}\,x_{n-1}\,x_n \lor \dots \lor \overline{x_1}\,x_2{\cdot}x_n \lor x_1\,x_2\cdots x_{n-1}\,x_n \quad .$$

$\square$

Corollary: Assuming that the probability of each miracle is $\frac{1}{2}$, and that they are done independently, the probability of **not** meeting the Dayenu requirement is $\frac{n+1}{2^n}$ and hence of meeting it is $1-\frac{n+1}{2^n}$, that happens to be, for $n=15$, $\frac{2047}{2048}=0.9995117\dots$.

Comments:

1. Surprisingly, what God actually did, performing *all* the miracles, is not part of the truth-set of the Dayenu Boolean function, since there is always at least one miracle that is **not** performed.

2. A faster proof, without Boolean logic, for getting the set of true-false vectors not satisfying the Dayenu function D_n (and hence proving the above Dayenu theorem), can be gotten by finding the set of all members of $\{T,F\}^n$ for which an F **never** (immediately) follows a T. Of course, this set is $\{F^n, F^{n-1}T, \dots, T^n\}$.

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Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .
