PROOF OF A CONJECTURE OF CHAN, ROBBINS, AND YUEN

Doron ZEILBERGER [1]

Abstract: Using the celebrated Morris Constant Term Identity, we deduce a recent conjecture of Chan, Robbins, and Yuen (math.CO/9810154), that asserts that the volume of a certain $n(n-1)/2$-dimensional polytope is given in terms of the product of the first $n-1$ Catalan numbers.

Chan, Robbins, and Yuen[CRY] conjectured that the cardinality of a certain set of triangular arrays $\mathcal{A}_n$ defined in pp. 6-7 of [CRY] equals the product of the first $n-1$ Catalan numbers. It is easy to see that their conjecture is equivalent to the following *constant term identity* (for any rational function $f(z)$ of a variable z, $CT_z f(z)$ is the coeff. of z^0 in the formal Laurent expansion of $f(z)$ (that always exists)):

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-2} \prod_{1 \le i < j \le n} (x_j - x_i)^{-1} = \prod_{i=1}^{n} \frac{1}{i+1} \binom{2i}{i} \quad . \qquad (CRY)$$

But this is just the special case $a=2$, $b=0$, $c=1/2$, of the *Morris Identity*[M] (where we made some trivial changes of discrete variables, and 'shadowed' it)

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b} \prod_{1 \le i < j \le n} (x_j - x_i)^{-2c} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)} \, . \qquad (Chip)$$

To show that the right side of $(Chip)$ reduces to the right side of (CRY) upon the specialization $a=2, b=0, c=1/2$, do the plugging in the former and call it M_n. Then manipulate the products to simplify M_n/M_{n-1}, and *then* use *Legendre's duplication formula* $\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\Gamma(1/2)/2^{2z-1}$ three times, and *voilà*, up pops the Catalan number $\binom{2n}{n}/(n+1)$. $\Box$

Remarks: 1. By converting the left side of $(Chip)$ into a contour integral, we get the same integrand as in the Selberg integral (with $a \to -a$, $b \to -b-1$, $c \to -c$). Aomoto's proof of the Selberg integral (SIAM J. Math. Anal. **18**(1987), 545-549) goes verbatim. **2.** Conjecture 2 in [CRY] follows in the same way, from (the obvious contour-integral analog of) Aomoto's extension of Selberg's integral. Introduce a new variable t, stick $CT_t t^{-k}$ in front of (CRY), and replace $(1-x_i)^{-2}$ by $(1-x_i)^{-1}(t + x_i/(1-x_i))$. **3.** Conjecture 3 follows in the same way from another specialization of $(Chip)$.

References

[CRY] Clara S. Chan, David P. Robbins, and David S. Yuen, *On the volume of a certain polytope*, `math.CO/9810154`.

[M] Walter Morris, *"Constant term identities for finite and affine root systems, conjectures and theorems"*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin, 1982.

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[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu` `http://www.math.temple.edu/~zeilberg/` . Nov. 17, 1998. Supported in part by the NSF.