In how many ways can I carry a total of n coins in my two pockets, and have the same amount in both pockets?

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In fond memory of Gert Almkvist (April 17, 1934- Nov. 24, 2018).*

Theorem 1: Let $a(n)$ be the number of ways of having a total of n coins in your two pockets (each of them either a penny, a nickel, a dime, or a quarter), so that the amounts in the pockets are identical, then

$$\sum_{n=0}^{\infty} a(n)t^n = \frac{P(t)}{Q(t)} \quad , where$$

$$P(t) = t^{54} + t^{53} + 3\,t^{52} + 4\,t^{51} + 9\,t^{50} + 15\,t^{49} + 25\,t^{48} + 37\,t^{47} + 54\,t^{46} + 76\,t^{45} + 101\,t^{44} + 128\,t^{43}$$
$$+158\,t^{42} + 190\,t^{41} + 226\,t^{40} + 256\,t^{39} + 290\,t^{38} + 318\,t^{37} + 353\,t^{36} + 372\,t^{35} + 394\,t^{34} + 405\,t^{33} + 425\,t^{32}$$
$$+431\,t^{31} + 439\,t^{30} + 438\,t^{29} + 448\,t^{28} + 448\,t^{27} + 448\,t^{26} + 438\,t^{25} + 439\,t^{24} + 431\,t^{23} + 425\,t^{22} + 405\,t^{21}$$
$$+394\,t^{20} + 372\,t^{19} + 353\,t^{18} + 318\,t^{17} + 290\,t^{16} + 256\,t^{15} + 226\,t^{14} + 190\,t^{13} + 158\,t^{12} + 128\,t^{11} + 101\,t^{10}$$
$$+76\,t^{9} + 54\,t^{8} + 37\,t^{7} + 25\,t^{6} + 15\,t^{5} + 9\,t^{4} + 4\,t^{3} + 3\,t^{2} + t + 1 \quad ,$$

and

$$Q(t) = (1-t)^7\,(1+t)^5\,\left(t^2+t+1\right)^3\left(t^2-t+1\right)^2\left(t^{12}+t^{11}+t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t+1\right)\cdot$$
$$\left(t^{12}-t^{11}+t^{10}-t^9+t^8-t^7+t^6-t^5+t^4-t^3+t^2-t+1\right)\left(t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t+1\right)\cdot$$
$$\left(t^6+t^5+t^4+t^3+t^2+t+1\right) \quad .$$

Furthermore $a(n)$ is a quasi-polynomial, and asymptotically,

$$a(n) = \frac{5821}{311351040}\,n^6 \,+\, O(n^5) \quad .$$

* Gert Almkvist was one of the most creative and original mathematicians that we have ever met. He was known, among his friends, as "the guy who generalized a mistake of Bourbaki" [see `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gert.html`], a master expositor (1989 Lester Ford award, joint with Bruce Berndt, and numerous articles in Swedish), the co-inventor of the Almkvist-Zeilberger algorithm, and a great authority on Calabi-Yao differential equations. In addition to his official affiliation with the University of Lund, he was the founder of the Institute of Algebraic Meditation, and many of his papers used it as his affiliation.

Finally

$$a(10^{100}) =$$

1869593883482772371661260550149439038327927216816105704994593883482772371661260
5501494390383279272213031310253532475754697976920199142421364643586865809088031
3102535324757546979769201991424213646436436428797539908651019762130873241984353
0954642065753176864287975399086510197621308732419843530954646363924141701919479
6972574750352528130305908083685861463639241417019194796972574750352528130305908
1017544132821910599688377466155243933021710799488577266355044132821910599688377
4661552439330217107995247753069975292197514419736641958864181086403308625530847
75306997529219751441973664195886418108640 35 .

The first 31 terms are:

$$1, 0, 4, 2, 12, 12, 34, 40, 85, 108, 190, 250, 394, 516, 762, 984, 1385, 1764, 2396,$$

$$2998, 3966, 4886, 6316, 7684, 9739, 11706, 14594, 17358, 21320, 25134, 30470 \quad .$$

As of Jan. 23, 2019, this sequence is not in the OEIS [Sl].

For analogous theorems where one can also have a half-dollar coin, and a half-dollar coin as well as a dollar coin, see the output files

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oEvenChange1b.txt`, and

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oEvenChange1c.txt` .

How did we get this amazing theorem?

let $A(n,m)$ be the number of ways of having n coins in your two pockets (with denominations $1, 5, 10, 25$) in such a way that the difference between the amount in the left pocket and the amount in the right pocket is m cents. Then, if we define

$$R(z,t) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A(n,m)\, t^n z^m \quad ,$$

then we have

$$R(z,t) \;=\; \frac{1}{(1-tz)(1-tz^5)(1-tz^{10})(1-tz^{25})} \cdot \frac{1}{(1-t/z)(1-t/z^5)(1-t/z^{10})(1-t/z^{25})} \quad .$$

This should be viewed as a *formal power series* in t whose coefficients are *Laurent polynomials* in z, and we are interested in extracting the coefficient of z^0. Now you ask Maple to kindly convert the above rational function into *partial fractions*, **with respect to the variable z**, getting something of the form

$$\frac{P_1(z,t)}{1-tz} + \frac{P_2(z,t)}{1-tz^5} + \frac{P_3(z,t)}{1-tz^{10}} + \frac{P_4(z,t)}{1-tz^{25}} + \frac{Q_1(z,t)}{z-t} + \frac{Q_2(z,t)}{z^5-t} + \frac{Q_3(z,t)}{z^{10}-t} + \frac{Q_4(z,t)}{z^{25}-t} \quad ,$$

for some explicit expressions in z, t, $P_1(z,t) \dots P_4(z,t)$, $Q_1(z,t) \dots Q_4(z,t)$, that Maple finds for you. These are **rational functions** in t but polynomials in z.

When we view them all as a formal power series in t, and take the coefficient of z^0, the Q's do not contribute anything, so the constant term, in z, is simply

$$P_1(0,t) + P_2(0,t) + P_3(0,t) + P_4(0,t) \quad .$$

This is implemented in the Maple package `EvenChange.txt` by procedure `GfPAB(P,z,t,A,B)` that finds the coefficient of z^0 of

$$\frac{P(z)}{\prod_{a \in A}(1 - z^a\, t) \prod_{b \in B}(1 - t/z^b)} \quad ,$$

for *any* polynomial P of z and *any* sets of positive integers A and B (so you can have different kinds of coins in each pocket, and also talk about the number of ways of doing it where the difference between the amounts is not necessarily 0).

The Maple package `EvenChange.txt` is available from the front of this article

`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/change.html` .

Since all the generating functions have denominators whose roots are roots of unity, the sequence of interest itself, $a(n)$, is a *quasi-polynomial*, albeit of a very large period. It is more efficient (still using partial fractions, this time with respect to t) to express it as a sum of quasi-polynomials of small periods. This is done via the procedure `GFtoQPS` that is lifted from the Maple package

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/PARTITIONS` ,

that accompanies [SiZ].

This is how we found so quickly $a(10^{100})$ and the leading asymptotics of $a(n)$ in Theorem 1.

Computing the generating functions $\psi_n(t)$ dear to Gert Almkvist, Cayley, and Sylvester

In his 1980 paper [A], Gert Almkvist was interested in the sequence of rational functions $\{\psi_n(t)\}$ that he defined as the constant term, in the variable z, of the rational function

$$\frac{(1+z)^2}{2\, z \prod_{i=0}^{n}(1 - t\, z^{n-2i})} \quad .$$

Using procedure `GfPAB` again, we got the following theorem. According to Almkvist [A], the cases $n = 2, 3, 4$ are due to Faa de Bruno and $n \leq 12$, except for $n = 11$ are due to Sylvester and Franklin.

Theorem 2:

- $n = 2$:

$$\psi_2(t) = \frac{1}{(1-t)^2\,(t+1)} \quad ,$$

and its n-th coefficient, $a_2(n)$, is asymptotically

$$a_2(n) \,=\, \frac{1}{2}n \,+\, O(1) \quad .$$

The first 31 terms (starting with $n = 0$) are:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 14, 14, 15, 15, 16 \quad .$$

This is A4536 [http://oeis.org/A004526] in [Sl].

- $n = 3$:

$$\psi_3(t) = \frac{t^2 - t + 1}{(1-t)^3\,(t+1)\,(t^2+1)} \quad ,$$

and its n-th coefficient, $a_3(n)$, is asymptotically

$$a_3(n) \,=\, \frac{1}{8}n^2 \,+\, O(n) \quad .$$

The first 31 terms (starting with $n = 0$) are:

$$1, 1, 2, 3, 5, 6, 8, 10, 13, 15, 18, 21, 25, 28, 32, 36, 41, 45, 50, 55, 61, 66, 72, 78, 85, 91, 98, 105, 113, 120, 128 \quad .$$

This is A1971 [http://oeis.org/A001971] in [Sl], that references [A].

- $n = 4$:

$$\psi_4(t) = \frac{t^2 - t + 1}{(t+1)\,(t^2+t+1)\,(-1+t)^4}$$

and its n-th coefficient, $a_4(n)$, is asymptotically

$$a_4(n) \,=\, \frac{1}{36}\,n^3 \,+\, O(n^2) \quad .$$

The first 31 terms (starting with $n = 0$) are:

$$1, 1, 3, 5, 8, 12, 18, 24, 33, 43, 55, 69, 86, 104, 126, 150, 177, 207, 241, 277, 318, 362, 410, 462, 519, 579, 645,$$

$$715, 790, 870, 956 \quad .$$

This is A1973 [http://oeis.org/A001973] in [Sl].

• $n=5$:

$$\psi_5(t) = \frac{t^{14}-t^{13}+2\,t^{12}+t^{11}+2\,t^{10}+3\,t^9+t^8+5\,t^7+t^6+3\,t^5+2\,t^4+t^3+2\,t^2-t+1}{(1-t)^5\,(t+1)^3\,(t^2+1)^2\,(t^2+t+1)\,(t^2-t+1)\,(t^4+1)} \quad ,$$

and its n-th coefficient, $a_5(n)$, is asymptotically

$$a_5(n) \;=\; \frac{23}{4608}\,n^4 \;+\; O(n^3) \;=\; \frac{23}{2^9\,3^2}\,n^4 \;+\; O(n^3) \quad .$$

The first 31 terms (starting with $n=0$) are:

$$1, 1, 3, 6, 12, 20, 32, 49, 73, 102, 141, 190, 252, 325, 414, 521, 649, 795, 967, 1165, 1394, 1651, 1944,$$
$$2275, 2649, 3061, 3523, 4035, 4604, 5225, 5910 \quad .$$

This is A1975 [`http://oeis.org/A001975`] in [Sl].

• $n=6$:

$$\psi_6(t) = \frac{t^{10}+t^8+3\,t^7+4\,t^6+4\,t^5+4\,t^4+3\,t^3+t^2+1}{(1-t)^6\,(t^2+1)\,(t^4+t^3+t^2+t+1)\,(t+1)^3\,(t^2+t+1)} \quad ,$$

and its n-th coefficient, $a_6(n)$, is asymptotically

$$a_6(n) \;=\; \frac{11}{14400}\,n^5 \;+\; O(n^4) \;=\; \frac{11}{2^6\,3^2\,5^2}\,n^5 \;+\; O(n^4) \quad .$$

The first 31 terms (starting with $n=0$) are:

$$1, 1, 4, 8, 18, 32, 58, 94, 151, 227, 338, 480, 676, 920, 1242, 1636, 2137, 2739, 3486, 4370, 5444, 6698, 8196, 9926,$$
$$11963, 14293, 17002, 20076, 23612, 27594, 32134 \quad .$$

This is A1977 [`http://oeis.org/A001977`] in [Sl].

• $n=7$:

$$\psi_7(t) = \frac{P_7(t)}{Q_7(t)} \quad ,$$

where

$$P_7(t) = t^{34}-t^{33}+3\,t^{32}+3\,t^{31}+7\,t^{30}+12\,t^{29}+16\,t^{28}+28\,t^{27}+33\,t^{26}+46\,t^{25}+56\,t^{24}+73\,t^{23}$$
$$+83\,t^{22}+90\,t^{21}+106\,t^{20}+109\,t^{19}+121\,t^{18}+110\,t^{17}+121\,t^{16}+109\,t^{15}+106\,t^{14}+90\,t^{13}+83\,t^{12}$$
$$+73\,t^{11}+56\,t^{10}+46\,t^9+33\,t^8+28\,t^7+16\,t^6+12\,t^5+7\,t^4+3\,t^3+3\,t^2-t+1 \quad ,$$

and

$$Q_7(t) = (1-t)^7\,(t+1)^5\,\left(t^2+1\right)^3\left(t^2+t+1\right)^2\left(t^2-t+1\right)^2 \cdot$$
$$\left(t^4+1\right)\left(t^4+t^3+t^2+t+1\right)\left(t^4-t^3+t^2-t+1\right)\left(t^4-t^2+1\right) \quad ,$$

and its n-th coefficient, $a_7(n)$, is, asymptotically

$$a_7(n) = \frac{841}{829440}\, n^6 + O(n^5) = \frac{29^2}{2^{12}3^4 5^2}\, n^6 + O(n^5) \quad .$$

The first 31 terms (starting with $n=0$) are:

$$1, 1, 4, 10, 24, 49, 94, 169, 289, 468, 734, 1117, 1656, 2385, 3370, 4672, 6375, 8550,$$

$$11322, 14800, 19138, 24460, 30982, 38882, 48417, 59779, 73316, 89291, 108108, 130053, 15564 \quad .$$

This is A1979 [`http://oeis.org/A001979`] in [Sl].

For the cases $8 \le n \le 18$, see the output file

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oEvenChange2b.txt` .

The case $n=8$ is A1981 [`http://oeis.org/A001981`] in [Sl]. As of Jan. 23, 2019, the cases $n=9$ and $n=10$ are not in the OEIS, and probably (we were too lazy to check) neither are the higher ones.

References

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