The C-finite Ansatz meets the Holonomic Ansatz

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VERY IMPORTANT

As in all our joint papers, the main point is not the article, but the accompanying Maple package, `CfiniteIntegral.txt`, that may be downloaded, free of charge, from the web-page of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfiniteI.html` ,

where the readers can also find sample input and output files, that they are welcome to extend using their own computers.

Preface

In a recent article, [Kim], symbolic summation, and quite a bit of human pre-processing, is used to evaluate certain integrals involving Chebyshev polynomials.

First, let's remark that the Chebyshev polynomials, like all *classical orthogonal polynomials*, belong to the **Holonomic Ansatz** ([Z1], beautifully, and very efficiently, implemented in [Kou]), and as such, *inter alia*, **every** identity in [Kim] (and in many other articles that are still published today) are *automatically provable*, and their *epistemological status* is the same as identities like $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$ or $134 \cdot 431 = 57754$. Could you imagine a paper published today (or even two thousands years ago), entitled "A new proof of the identity $134 \cdot 431 = 57754$"?,

Introduction

But the Chebyshev polynomials are not 'just' *holonomic*, they belong to the more restricted class of C-finite polynomial sequences, and hence belong to the C-finite ansatz ([Z2]), and as such have nice closure properties. It turns out that one can *interface* the C-finite ansatz and the *holonomic* ansatz, and borrow from the latter the powerful, and not-as-well-known-as-it-should-be

"**the (continuous) Almkvist-Zeilberger algorithm**",

described in [AZ], and implemented in the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt` .

We combined all the procedures from the above Maple package and the Maple package that accompanies [Z2], and created a new, self-contained, Maple package,

`http:/www.math.rutgers.edu/~zeilberg/tokhniot/CfiniteIntegral.txt` , that can **automatically** prove *every* identity in [Kim], and many, **far deeper**, identities.

We should note that [Kim] also uses computer algebra methods, but those involving *summation*, and spends quite a lot of *human effort* to go from integration to summation. With the Almkvist-Zeilberger algorithm, one can proceed directly, as follows.

Using the Continuous Almkvist-Zeilberger algorithm in order to Evaluate (Symbolically!) Integrals of Powers of [in particular] Chebyshev Polynomials

Suppose that you want to study a sequence of the form

$$a(n) := \int_{\alpha}^{\beta} P_n(x)\, K(x)\, dx \quad ,$$

for some "nice" *kernel*, $K(x)$, and a sequence $P_n(x)$ of C-finite polynomials, i.e. given by a recurrence

$$P_n(x) = \sum_{i=1}^{L} p_i(x) P_{n-i}(x) \quad ,$$

for some positive integer, L, (the *order*) and polynomials $p_i(x)$ ($i = 1, \dots, L$), subject to *initial conditions*

$$P_0(x) = q_0(x), \dots, P_{L-1}(x) = q_{L-1}(x) \quad ,$$

for some polynomials $q_0(x), \dots, q_{L-1}(x)$.

What's nice about the C-finite ansatz is that once you know that $\{P_n(x)\}$ is C-finite, the same is true for $\{P_n(x)^r\}$ for any positive integer r, and also the sequence $P_n(x)P_n^*(x)$ (where for any polynomial $a(x)$, $a^*(x) = x^d a(1/x)$ (where d is the degree of $a(x)$), is its "reverse"), and many other related sequences, and one can fully automatically (and very fast) find C-finite representations for them (see [Z2], and [KP] (a true **masterpiece**!)).

Once we have such a C-finite polynomial sequence, the *ordinary* generating function

$$R(x,t) := \sum_{n=0}^{\infty} P_n(x) t^n \quad ,$$

is a certain *rational function*, $R(x,t)$, of the variables x and t, that can be easily, and quickly, found automatically. Hence the (ordinary) generating function of the sequence $a(n)$, let's call it $f(t)$,

$$f(t) := \sum_{n=0}^{\infty} a(n)\, t^n \quad ,$$

can be expressed as

$$f(t) = \int_{\alpha}^{\beta} R(x,t) K(x)\, dx \quad .$$

The continuous Almkvist-Zeilberger algorithm produces a *linear differential operator* with *polynomial* coefficients, $\mathcal{P}(t, \frac{d}{dt})$, and a *certificate* (a rational function times the integrand), let's call it $C(x,t)$, such that

$$\mathcal{P}(t, \frac{d}{dt})\,[\,R(x,t)K(x)\,] \;=\; \frac{d}{dx}C(x,t) \quad .$$

Integrating with respect to x, from $x=\alpha$ to $x=\beta$, we get

$$\mathcal{P}(t, \frac{d}{dt})\left[\int_\alpha^\beta R(x,t)K(x)\,dx\right] = \int_\alpha^\beta \left(\frac{d}{dx}C(x,t)\right) \quad .$$

Hence, by the *Fundamental Theorem of Calculus*, $f(t)$ satisfies an *inhomogeneous* ordinary differential equation with polynomial coefficients

$$\mathcal{P}(t, \frac{d}{dt})f(t) = C(\beta,t) - C(\alpha,t) \quad . \qquad (DiffEq)$$

It is readily seen that the right-hand-side is a rational function in t. The differential equation $(DiffEq)$ immediately implies a *linear (inhomogeneous) recurrence equation with polynomial coefficients* for the actual coefficients, namely, our original $a(n)$, from which, using standard methods, one can get a *homogeneous* linear recurrence equation with polynomial coefficients, that together with the *initial conditions*, that can be easily computed, constitutes a full **description** of the sequence $a(n)$, that enables us to compute the sequence to as-many-as-desired terms.

In fact, since we have the *theoretical guarantee* that such a description **exists** we can even skip the above steps, and resort to *pure guessing*!

As we have mentioned at the beginning, the above-mentioned web-page

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfiniteI.html` ,

contains numerous sample input and output files (including fully automatic proofs of the results of [Kim]), that readers can extend to their heart's content.

References

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