The Amazing 3^n Theorem and its even more Amazing Proof
[Discovered by Xavier Viennot and his École Bordelaise gang]

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Pour mon Cher ami, Guru Xavier G. VIENNOT

Definition: A *xavier* is a 2D tower of domino pieces where all the domino pieces at the bottom floor are contiguous (i.e. no gaps), and every domino-piece at a higher floor is placed in such a way that its middle-line is aligned with the left-end and/or right-end of (one or two) pieces at the floor right below it.

Theorem [Gouyou-Beauchamps & Viennot [GV], announced in the *historic* article [V], p.233]:
The number of xaviers with $n+1$ pieces is 3^n.

Proof [Bétréma & Penaud[BeP], briefly sketched ("pictorially") on pp. 47-48 of Mireille Bousquet-Mélou's great *habilitation* thesis[Bo1], and section 3.4.2 in yet-another *historic* article [Bo2], and also reproduced on p. 81 of the Flajolet-Sedgewick [FS] *bible*].

Define the weight of a xavier, x, by: $weight(x) := z^{NumberOfPiecesOf\ x}$.

Let a *pyramid* be a xavier whose bottom floor only has one piece, and let a *half-pyramid* be a pyramid where no piece is strictly to the left of the bottom piece.

Every half-pyramid is either

(i) the singleton xavier (let's call it Z)

(ii) a half-pyramid where all the pieces of the higher floors are **strictly** to the **right** of the bottom piece

(iii) a half-pyramid where at least one of the leftmost pieces of a non-bottom floor is aligned with the left-end of the bottom piece.

Let's call the set of half-pyramids $\mathcal{H}$.

The set of members of (ii) are in bijection with the Cartesian product $\{Z\} \times \mathcal{H}$ (just remove the bottom piece, and to go back, move the half-pyramid one unit to the right and stick Z at the bottom.

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Let h be a half-pyramid of case (iii). We shall map it to a triple $[Z, h_1, h_2]$ where h_1 and h_2 are smaller half-pyramids.

Let Z_2 be the lowest piece strictly above the bottom floor piece (that we call Z) that is aligned with it. If there is nothing above Z_2, let h_1 be the half-pyramid obtained from h by removing both Z and Z_2, and let $h_2 = Z_2$. Otherwise, look at the lowest piece "above" Z_2 (either touching it or not), and then, keep adding to h_2 by looking at the pieces above this newly-acquired piece, and continue recursively, for each newly-acquired piece, finding those pieces above them (if they exist). Continue until you can't find anything more. h_1 is the half-pyramid obtained by removing the bottom Z and all the pieces of h_2, and (the final version of) h_2 consists of the above pieces, at the same horizontal locations, but closing the vertical gaps (by dropping pieces), if necessary.

So we have a mapping

$$\mathcal{H} \rightarrow \{Z\} \bigcup \{Z\} \times \mathcal{H} \bigcup \{Z\} \times \mathcal{H} \times \mathcal{H} \quad .$$

It is easy to see that it is a bijection. For a member $[Z, h_1, h_2]$ stick Z below the bottom of h_1, one unit to the left, and then place h_2 completely above h_1, aligning the left of h_2 with Z. Starting with the bottom piece of h_2, and working your way up, and then left-to-right, one-by-one, construct h, dynamically, by "dropping" each piece of h_2 until it "lends" on the current h. The final outcome is h.

So we can write

$$\mathcal{H} \equiv \{Z\} \bigcup \{Z\} \times \mathcal{H} \bigcup \{Z\} \times \mathcal{H} \times \mathcal{H} \quad .$$

Let $H = H(z)$ be the sum of the weights of the members of $\mathcal{H}$. By taking weights, we get the *quadratic* equation

$$H = z + zH + zH^2 \quad , \qquad (Jean)$$

and if you know *advanced*, middle-school, mathematics (or were a Babylonian scholar 3000 years ago), and know how to solve, via radicals, a quadratic equation, you would be able to get an "explicit" expression for H, and by `taylor`ing, you could get the first one hundred (or whatever) coefficients, and by *peeking* at Sloane, you would find out that the enumerating sequence is the sequence of *Motzkin numbers*. But this is besides (our!) point. We want to keep everything elementary (no square-roots, (and definitely no tayloring) please!)

What about pyramids? Let $\mathcal{P}$ be the set of pyramids.

Let p be a pyramid. Of course it may be a half-pyramid, but if not we will map it into a pair $[h_1, p_1]$ where h_1 is a half-pyramid and p_1 is a smaller pyramid.

If p is not already a half-pyramid, there must be, looking from bottom-to-top, a *lowest* piece whose left-end is *strictly* to the left (of course, one unit exactly) of the left-end of the bottom piece of p. Let's call it Z_1. If there is nothing above Z_1, that's our p_1. Otherwise, as above, find all the pieces above it, and having found them, those above the latter (once again, either touching or not), and continue recursively until there is nothing more. After removing all these pieces we are left with a half-pyramid, let's call it h_1, and the removed pieces, after closing, if necessary, vertical gaps, is a brand-new pyramid, let's call it p_1. Once again it is obvious that you can get p back from $[h_1, p_1]$, by shifting p_1 one unit to the left of the bottom-piece of h_1 and "dropping" the pieces, one-by-one, from bottom to top, and from left-to-right. So we have

$$\mathcal{P} \equiv \mathcal{H} \bigcup \mathcal{H} \times \mathcal{P} \quad .$$

Taking weights, and letting $P = P(z)$ be the sum of the weights of all members of $\mathcal{P}$, we have the *linear* equation:

$$P = H + HP \quad , \qquad (Jean - Guy)$$

that implies $P = H/(1 - H)$, and if you were stupid enough to "solve" for H above, and plug-it-in into $P = H/(1 - H)$, you would get an even uglier expression for P, in terms of the same annoying radical sign (reproduced in [FS], eq. (86), but I don't like radicals, so I will spare you!)

Finally, let $\mathcal{X}$ be the set of xaviers, and let x be a typical xavier. If the bottom floor only has one piece, then it is a pyramid. Otherwise the bottom consists of at least two adjacent pieces. Let's ignore, for now, the rightmost piece of the bottom floor, and look at the remaining pieces. These will form the bottom floor of a new, smaller, xavier, let's call it x_1. To get the rest of x_1, once again, let's look at all the pieces above them, and those above the latter, and continue recursively. It is obvious that when you remove all these pieces from x you would get a half-pyramid (whose bottom was the rightmost piece of x), and x_1 is formed by closing all the vertical gaps, getting a brand-new xavier, whose bottom floor has one-less-piece than the bottom floor of x. This gives the weight-preserving bijection

$$\mathcal{X} \leftrightarrow \mathcal{P} \bigcup \mathcal{H} \times \mathcal{X} \quad ,$$

and taking weights, letting $X = X(z)$ be the sum of the weights of the members of $\mathcal{X}$, we get yet another *linear* equation

$$X = P + HX \quad , \qquad (Xavier)$$

that implies that $X = P/(1 - H)$, that, in turn, using the previously established $P = H/(1 - H)$, imples that

$$X = \frac{H}{(1 - H)^2} \quad .$$

Now, if you are a high-school-algebra whiz, you can take the "explicit" expression for H alluded to above (but intentionally suppressed here) and "do the algebra", and *mirabile dictu*, you would get

$$X(z) = \frac{z}{1 - 3z} \quad .$$

That, in turn, implies, by taking the coefficient of z^{n+1} in $X(z)$, that the number of xaviers with $n + 1$ pieces is 3^n. But this is way too advanced! Here is an alternative, elementary, proof, not using the advanced quadratic formula and tedious manipulations with radicals. We first multiply top and bottom by z, getting

$$X = \frac{zH}{z(1 - H)^2} \quad .$$

Eq. $(Jean)$ implies $zH^2 = H - z - zH$, and using $(1-H)^2 = 1 - 2H + H^2$, we get:

$$X = \frac{zH}{z(1-H)^2} = \frac{zH}{z - 2zH + zH^2} = \frac{zH}{z - 2zH + (H - z - zH)} = \frac{zH}{H - 3zH} = \frac{zH}{(1-3z)H} = \frac{z}{1-3z} \quad ,$$

et voilà, a *truly* elementary proof! Or is it? Not really! Since it uses the very advanced, and counter-intuitive, notion of *subtraction*. Here is an even better proof, only using addition, of the equivalent fact

$$X = z + 3zX \quad .$$

Indeed, thanks to $(Jean)$:

$$X = \frac{H}{(1-H)^2} = \frac{z + zH + zH^2}{(1-H)^2} = \frac{z}{(1-H)^2} + z\frac{H}{(1-H)^2} + \frac{zH^2}{(1-H)^2} = \frac{z}{(1-H)^2} + zX + \frac{zH^2}{(1-H)^2} \quad .$$

Now using the deep identity $\frac{1}{1-H} = 1 + \frac{H}{1-H}$ twice yields

$$\frac{1}{(1-H)^2} = 1 + \frac{H}{1-H} + \frac{H}{(1-H)^2} \quad .$$

So

$$X = \frac{z}{(1-H)^2} + zX + \frac{zH^2}{(1-H)^2} = z + z\frac{H}{1-H} + z\frac{H}{(1-H)^2} + zX + \frac{zH^2}{(1-H)^2} = z + z\frac{H}{1-H} + zX + zX + z\frac{H^2}{(1-H)^2}$$

$$= z + 2zX + z\left(\frac{H}{1-H} + \frac{H^2}{(1-H)^2}\right) = z + 2zX + z\frac{H}{(1-H)^2} = z + 2zX + zX = z + 3zX \quad \square \quad .$$

I claimed above that the above algebraic proof is *minus-sign-free*, but one may object that in the *denominators* we have $1 - H$ and $(1-H)^2$ all over the place, **but** $\frac{1}{1-H}$ is just shorthand for the minus-sign-free geometric series $\sum_{i=0}^{\infty} H^i$ and $\frac{H}{1-H}$ is $\sum_{i=1}^{\infty} H^i$ and $\frac{H}{(1-H)^2}$ is short for $\left(\sum_{i_1=0}^{\infty} H^{i_1}\right) H \left(\sum_{i_2=0}^{\infty} H^{i_2}\right)$.

A Natural bijection between xaviers with n+1 pieces and words of length n in the alphabet $\{-1, 0, 1\}$

The *first* proof of the 3^n theorem, in [GV], was via such a bijection. Alas, it was rather complicated and *ad hoc*. By taking the Bétréma & Penaud constructions above, and by reverse-engineering the above simple algebraic proof, one can easily get a bijection, that may or may not be the same, or equivalent to, the original bijection, but is much easier to state.

The first step is to note that by iterating the $\mathcal{P} \equiv \mathcal{H} \bigcup \mathcal{H} \times \mathcal{P}$ bijection, there is a natural bijection between $\mathcal{P}$ and non-empty lists of half-pyramids. By iterating $\mathcal{X} \equiv \mathcal{P} \bigcup \mathcal{H} \times \mathcal{X}$ we get that any xavier corresponds uniquely to a creature of the form

$$[[h_1, \dots, h_a], h_0, [f_1, \dots, f_b]] \quad ,$$

where $a, b \geq 0$ and $h_1, \dots, h_a, h_0, f_1, \dots, f_b$ are all half-pyramids. From this we can extract a letter of the alphabet $\{-1, 0, 1\}$ and such a creature with one less piece. Here goes.

If $a = b = 0$ and h_0 is the single-piece Z, then it is rock-bottom, and the output is the empty-word.

Apply the first Bétréma-Penaud mapping (let's call it BP_1) to h_0. If $BP_1(h_0)$ is in $\{Z\} \times \mathcal{H}$, i.e. of the form $[Z, h'_0]$ for some half-pyramid h'_0, then the output is the letter 0 and the smaller creature is:

$$[[h_1, \ldots, h_a], h'_0, [f_1, \ldots, f_b]] \quad .$$

If $BP_1(h_0)$ is in $\{Z\} \times \mathcal{H} \times \mathcal{H}$, i.e. of the form $[Z, h'_0, h''_0]$ for some half-pyramids h'_0, h''_0, then the output is the letter 1 and the smaller creature is:

$$[[h_1, \ldots, h_a, h'_0], h''_0, [f_1, \ldots, f_b]] \quad .$$

If $BP_1(h_0)$ is the one-piece half-pyramid Z, then:

if $a > 0$ the output is the letter -1 and the smaller creature is:

$$[[h_1, \ldots, h_{a-1}], h_a, [f_1, \ldots, f_b]] \quad ,$$

while if $a = 0$, then we must have $b > 0$ and the output is the letter 1 (again) and the smaller creature

$$[[], f_1, [f_2, \ldots, f_b]] \quad .$$

By iterating this mapping we get a natural (recursive) mapping from our creatures (that are in natural bijection with xaviers) and words in the alphabet $\{-1, 0, 1\}$. The readers are welcome to formulate the inverse mapping.

As with the original bijection in [GV], once again we have a fast way to *randomly generate* a given xavier with $n + 1$ pieces. Just generate a random string of length n in $\{-1, 0, 1\}$, and apply the reverse of the above mapping. It is easy to see that pyramids correspond to words all whose partial sums are non-negative, and the half-pyramids to such words for which, in addition, the total sum is exactly zero.

The Maple package BORDELAISE

Everything here, and much more, is implemented in the Maple package `BORDELAISE`, downloadable directly from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/BORDELAISE` .

The front of this article,

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bordelaise.html` ,

contains sample input and output files, some with nice pictures.

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