CHU'S 1303 IDENTITY IMPLIES BOMBIERI'S 1990 NORM-INEQUALITY [Via An Identity of Beauzamy and Dégot]

(Appeared in the *Amer. Math. Monthly* Nov. 1994.)

Doron ZEILBERGER[1]

Blessed are the meek: for they shall inherit the earth (Matthew V.5)

Inequalities are deep, while *equalities* are shallow. Nevertheless, it sometimes happens that a deep inequality, **A**, follows from a mere *equality* **B**, which, in turn, follows from a more general, and *trivial*[2] identity **C**.

In this note we demonstrate this, following [3], with **A**:= Bombieri's norm inequality[2][3], **B**:= an identity of Reznick[5], and **C** := an identity of Beauzamy and Dégot[3]. This exposition differs from the original only in the punch line: I give a 1-line proof of **C**, using Chu's identity.

Let $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ be two polynomials in n variables:

$$P = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \quad , \quad Q = \sum_{i_1,\ldots,i_n \geq 0} b_{i_1,\ldots,i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \quad .$$

The *Bombieri inner product*[2] is defined by

$$[P, Q] := \sum_{i_1,\ldots,i_n \geq 0} (i_1! \ldots i_n!) \cdot a_{i_1,\ldots,i_n} b_{i_1,\ldots,i_n} \quad ,$$

and the *Bombieri norm*, by: $\|P\| := \sqrt{[P,P]}$.

Bombieri's Inequality A: Let P and Q be any *homogeneous* polynomials in $(x_1, \ldots, x_n)$, then

$$\|PQ\| \geq \|P\| \|Q\| \quad .$$

In order to state **B** and **C**, we need to introduce the following notation. $D_i := \frac{\partial}{\partial x_i}$, $(i = 1, \ldots, n)$, $P^{(i_1,\ldots,i_n)} := D_1^{i_1} \ldots D_n^{i_n} P$, and for any polynomial $A(x_1, \ldots, x_n)$, $A(D_1, \ldots, D_n)$ denotes the linear partial differential operator with constant coefficients obtained by replacing x_i by D_i.

A follows almost immediately from([5][3]):

Reznick's Identity B: For any polynomials P, Q in n variables:

$$\|PQ\|^2 = \sum_{i_1,\ldots,i_n \geq 0} \frac{\|P^{(i_1,\ldots,i_n)}(D_1, \ldots, D_n) Q(x_1, \ldots, x_n)\|^2}{i_1! \cdot \ldots \cdot i_n!} \quad .$$

Beauzamy and Dégot's Identity C: For any polynomials P,Q,R,S in n variables:

---

1 Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu` . Supported in part by the NSF. This note was written while the author was on leave (Fall 1993) at the Institute for Advanced Study, Princeton. I would like to thank Don Knuth for a helpful suggestion.

2 Trivial to verify, not to conceive!

3 It was needed by Beauzamy and Enflo in their research on deep questions on Banach spaces. It also turned out to have far reaching applications to computer algebra![1].

$$[PQ,RS] = \sum_{i_1,\ldots,i_n \geq 0} \frac{[R^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)Q(x_1,\ldots,x_n), P^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)S(x_1,\ldots,x_n)]}{(i_1!\ldots i_n!)} \quad .$$

Proof of B $\Rightarrow$ A: Pick the terms for which $i_1+\ldots+i_n$ equals the (total) degree of P, let's call it p, and note that for those $(i_1,\ldots,i_n)$, $P^{(i_1,\ldots,i_n)}(x_1,\ldots,x_n) = (i_1!\ldots i_n!)a_{i_1,\ldots,i_n}$, so

$$\sum_{i_1+\ldots+i_n=p} \frac{\|P^{(i_1,\ldots,i_n)}(D_1,\ldots,D_n)Q(x_1,\ldots,x_n)\|^2}{i_1!\cdot\ldots\cdot i_n!} = \sum_{i_1+\ldots+i_n=p} \|a_{i_1,\ldots,i_n}Q(x_1,\ldots,x_n)\|^2\cdot(i_1!\cdot\ldots\cdot i_n!)$$

$$= \left(\sum_{i_1+\ldots+i_n=p} (a_{i_1,\ldots,i_n})^2\cdot(i_1!\cdot\ldots\cdot i_n!)\right)\|Q(x_1,\ldots,x_n)\|^2 = \|P\|^2\|Q\|^2 \quad .$$

Proof of C $\Rightarrow$ B: Take $R=P$ and $S=Q$.

Proof of C: Both sides are linear in P, in Q, in R, and in S, so it suffices to take them all to be typical monomials, ($P = x_1^{p_1}\cdot\ldots\cdot x_n^{p_n}$, and similarly for Q,R, and S), for which the assertion follows immediately by applying Chu's[4] identity[4]

$$\sum_{i\geq 0}\binom{r}{i}\binom{s}{p-i} = \binom{r+s}{p} \quad ,$$

to $r=r_t, s=s_t, p=p_t, (t=1\ldots n)$, (using i_t for i), and taking their product. Q.E.D.

References

1. B. Beauzamy, *Products of polynomials and a priori estimates for coefficients in polynomial decompositions: A sharp result*, J. Symbolic Computation **13**(1992), 463-472.
2. B. Beauzamy, E. Bombieri, P. Enflo, and H.L. Montgomery, *Products of polynomials in many variables*, J. Number Theory **36** (1990), 219-245.
3. B. Beauzamy and J. Dégot, *Differential Identities*, I.C.M., Paris, preprint.
4. Chu Chi-kie, manuscript, 1303, China. (See J. Needham, Science and Civilization in China, v. 3, Cambridge University Press, New York, 1959.)
5. B. Reznick, *An inequality for products of polynomials*, Proc. Amer. Math. Soc. **117**(1993), 1063-1073.

Nov. 3, 1993 ; Revised: April 5, 1994.

Added Sept. 24, 2017: Kai-Liang Lin just informed that the so-called Bombieri's inequality should be called Neuberger's inequality, as it was discovered, in 1974, by John M. Neuberger. John M. Neuberger, "Norm of symmetric product compared with norm of tensor product", Linear and Multilinear Algebra **2**(1974), 115-121.

[4] Rediscovered in the 18th century by Vandermonde. Proved by counting, in two different ways, the number of ways of picking p lucky winners out of a set of r boys and s girls.