On the Average Maximal Number of Balls in a Bin Resulting from Throwing r Balls into n Bins T times

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Abstract. We use the *holonomic ansatz* to estimate the asymptotic behavior, in T, of the average maximal number of balls in a bin that is obtained when one throws uniformly at random (without replacement) r balls into n bins, T times. Our approach works, in principle, for any fixed n and r. We were able to do the cases $(n, r) = (2, 1), (3, 1), (4, 1), (4, 2)$, but things get too complicated for larger values of n and r. We are pledging a \$150 donation to the OEIS for an explicit expression, (in terms of n, r, and π) for the constant $C_{n,r}$ such that that average equals $\frac{n}{r}T + C_{n,r}\sqrt{T} + O(1/\sqrt{T})$.

Added May 23, 2019: Marcus Michelen came as close as possible to meeting our challenge. See his interesting note *A Short Note on the Average Maximal Number of Balls in a Bin*, `https://arxiv.org/abs/1905.08933` (submitted May 22, 2019). Before that Brendan McKay sketched a version of this argument, see
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/binsFeedback.html` .
A donation of \$150 to the OEIS, in honor of Marcus Michelen has been made.

Throwing r balls into n bins T times

The following problem arises in computer systems where jobs get to execute, either completely or partially, at different servers. Specifically, the load distribution across servers can be studied using balls and bins problem.

Suppose that you have n bins, and in each round, you throw r balls such that each ball lands in a different bin. We assume that each of the $\binom{n}{r}$ possibilities are equally likely.

After T rounds, you look at the maximum occupancy amongst the n bins. Let $U(n, r; T)$ be that random variable. The smallest value it can take is, of course, $\frac{r}{n}T$, where all the Tr balls are equally shared by all the n bins, while the largest value it can take is T, where one lucky bin gets a ball at each of the T rounds.

Let $A(n, r; T)$ be the expectation of $U(n, r; T)$ minus $\frac{r}{n}T$. Our questions are:

• For specified n and r, can we compute $A(n, r; T)$ **exactly** for very large values of T?

• For specified n and r, can we determine the asymptotic behavior (in T), of $A(n, r; T)$?

The answer to the first question is "**of course!**", but only *in principle.* If $P(n, r, T; a_1, \dots, a_n)$ is the probability that after T rounds the occupancy of the n bins are $a_1, \dots, a_n$ respectively, then

$$P(n, r, T; a_1, \dots, a_n) = \binom{n}{r}^{-T} \cdot Coefficient\ Of\ x_1^{a_1} \cdots x_n^{a_n} \quad in \quad e_r(x_1, \dots, x_n)^T \quad ,$$

where $e_r(x_1, \dots, x_n)$ is the *elementary symmetric function* of degree r in $x_1, \dots, x_n$, that may be defined by

$$e_r(x_1,\dots,x_n) = \sum_{1\le i_1<i_2<\dots<i_r\le n} x_{i_1}\cdots x_{i_r} \quad .$$

Since we can compute $P(n,r,T;a_1,\dots,a_n)$, we can also compute

$$Pr(U(n,r,T)\le m) \ = \sum_{a_1,\dots,a_n\le m} P(n,r,T;a_1,\dots,a_n) \quad ,$$

from which we can compute the probability mass function, and from that, in turn, the expectation. Unfortunately, *in practice*, this will not work for very large T, even for small n and r.

However, using *Wilf-Zeilberger algorithmic proof theory* [WZ] we know **for sure**, that for any specific n and r, our sequence of interest, $A(n,r;T), 1\le T<\infty$ satisfies a *linear recurrence equation with polynomial coefficients* with respect to T. While there exist algorithms to find these recurrences from first principle, they are very slow. It is much more efficient to compute the first few terms (using the above brute-force method), and then *guess* a linear recurrence, that we can justify rigorously *a posteriori*, since we know that such a recurrence exists, and we can theoretically bound its order. We found such recurrences for the cases $(n,r)=(2,1),(3,1),(4,1),(4,2)$.

Here there are:

Proposition 1: $A(T)=A(2,1;T)$ satisfies the following linear recurrence

`A(T) = 1/(T-1)*A(T-1)+A(T-2)`, with initial conditions $A(1)=1/2, A(2)=1/2$.

In fact, it is very easy to get a **closed-form** expression for $A(2T-1)$, even without WZ theory. Since

$$A(2T-1) \ = \ -T \ + \ \frac{1}{2} \ + \ \frac{1}{2^{2T-2}}\sum_{k=T}^{2T-1}\binom{2T-1}{T}T \quad ,$$

it is readily seen that $A(2T-1)$ is given explicitly by the following (humanly-generated!) proposition.

Proposition 1'

$$A(2T-1) \ = \ \frac{(2T-1)!}{2^{2T-1}(T-1)!^2} \quad ,$$

that implies, via Stirling's formula, that

$$A(T) = \frac{1}{\sqrt{2\pi}}\sqrt{T} + O(\frac{1}{\sqrt{T}}) \quad .$$

Proposition 2: $A(T)=A(3,1;T)$ satisfies the following fifth-order linear recurrence equation

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A(T) = 2/3*(7*T**4-32*T**3+36*T**2+19*T-36)/(7*T**2-25*T+20)/(T-1)**2*A(T-1) +1/3*(7*T**5-
53*T**4+131*T**3-55*T**2-206*T+200)/(T-2)/(7*T**2-25*T+20)/(T-1)**2*A(T-2) +1/3*(21*T**4-
```

```
180*T**3+519*T**2-544*T+160)/(7*T**2-25*T+20)/(T-1)**2*A(T-3)  -2/3*(T-3)*(7*T**3-
32*T**2+29*T+2)/(7*T**2-25*T+20)/(T-1)**2*A(T-4) -1/3*(T-3)*(T-4)*(7*T**2-11*T+2)/(7*T**2-
25*T+20)/(T-1)**2*A(T-5) ,
```

with initial conditions $A(1) = 2/3$, $A(2) = 2/3$, $A(3) = 8/9$, $A(4) = 28/27$, $A(5) = 10/9$.

Proposition 3: $A(T) = A(4, 1; T)$ satisfies a ninth-order linear recurrence equation that can be found here: `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oBINnr3.txt` .

Proposition 4: $A(T) = A(4, 2; T)$ satisfies an eighth-order linear recurrence equation that can be found here: `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oBINnr4.txt` .

It also follows (at least in the above cases, but probably can be proved in general), that for each n and r there exists a constant $C_{n,r}$ such that, asymptotically

$$A(n, r; T) = C_{n,r}\sqrt{T} + O(\frac{1}{\sqrt{T}}) \quad .$$

By using the recurrences to compute many values, we were able to derive the following estimates

$$C_{2,1} = 0.3989\ldots \quad , \quad C_{3,1} = 0.489\ldots \quad , \quad C_{4,1} = 0.516\ldots \quad , \quad C_{4,2} = 0.59430\ldots \quad .$$

(Note that the value for $C_{2,1}$ agrees with the human-generated *exact value* given by Proposition $1'$, $(2\pi)^{-1/2} = 0.398942280401\ldots$).

Using the algorithm in [Z], it is possible to get higher-order asymptotics, that usually enable a much more precise estimate of the leading constant of the asymptotics. To our disappointment, this does not seem to be the case for this problem, hence we do not bother to state this more refined asymptotics. Our computers are not large enough to handle larger values of n and r, even though we do know that such a recurrence exists for every choice of n and r.

The Maple package. The accompanying Maple package `BINnr.txt` is available from the front of this article:
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bins.html` .

Estimating $C_{n,r}$ for arbitrary n and r

Using heuristic arguments, Behrouzi-Far and Soljanin obtained estimates for $C_{n,r}$, that do not quite agree with the above values for small n and r, but are hopefully close to the exact values for larger n and r. It is

$$(r/n)\,\sqrt{\pi\, log(n)}\,\log(n/r) \quad .$$

Conclusion. While the results of this paper may be disappointing to people who are interested in this problem for large n and r, it is interesting from the point of view of *experimental mathematics*. By using the starting values of the sequence of interest, $A(n, r; T)$, as 'training data', that were computed by brute force, the computer 'learned' the linear recurrence, by fitting it into an *ansatz* (see [KP], chapter 7) and this enabled it to *predict* (exactly!) the values for very large T, that would be impossible to compute, in practice, by the naive method.

Encore: An asymptotic challenge

Our quantity of interest, $A(n,r;T)$, is expressible as a multi-variable contour integral, hence it may be possible to find the asymptotic constant, $C_{n,r}$, as a closed-form expression in terms of n and r and π. It is also expressible, via the multinomial theorem, as a huge multi-sum featuring multinomial coefficients. The special case $r=1$ 'only' has $n-1$ $\sum$-signs, and may be not too difficult, but the general r seems harder. The authors of the modern classic [PW], or one of their disciples, may rise to the challenge of finding that expression.

We are pledging a \$50 donation to the OEIS in honor of the first person to find a closed-form expression, in terms of n and π, for the case $r=1$, i.e. for $C_{n,1}$, and a further \$100 donation for the general case, i.e. for $C_{n,r}$.

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