How Much Should a 19th-Century French Bastard Inherit

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Dedicated to Gerry Ladas on his 60th Birthday.

Abstract: Catalan's formula, for the portion of the inheritance that a legitimate child of a 19th-century deceased French gentleman should receive, is given a new proof (using Difference Operators), and generalized. Another, more computationally efficient, formula is also derived.

Catalan's Formula

Chrystal's classical text[Ch] (II, p. 416) contains the following charming exercise, due to Catalan[Ca].

By French law an illegitimate child receives one-third of the portion of the inheritance that he would have received had he been legitimate. If there are l legitimate and n illegitimate children, show that the portion of inheritance 1 *due to a legitimate child is*

$$\frac{1}{l} - \frac{n}{3l(l+1)} + \frac{n(n-1)}{3^2 l(l+1)(l+2)} - \ldots \pm \frac{n(n-1)\ldots 2.1}{3^n l(l+1)\ldots(l+n)} \quad .$$

Let's call the portion due to a legitimate child $a(l,n)$. If the condition would have been that each illegitimate child receives one-third of what a legitimate child is to receive, this would have yielded the simple algebraic equation $la(l,n) + n(1/3)a(l,n) = 1$ and hence $a(l,n) = 3/(3l+n)$. But since the bastard is to receive one-third of *what he would have received had he been legitimate*, we have a *difference* (or recurrence) equation:

$$la(l,n) + (n/3)a(l+1, n-1) = 1 \quad . \qquad (*)$$

Introducing the shift operators $Lf(l,n) := f(l+1,n)$ and $Nf(l,n) = f(l,n+1)$, this can be rewritten:

$$(1 + \frac{n}{3l} LN^{-1})a(l,n) = \frac{1}{l} \quad ,$$

yielding:

$$a(l,n) = (1 + \frac{n}{3l}LN^{-1})^{-1}\frac{1}{l} = \sum_{r=0}^{\infty}(-\frac{n}{3l}LN^{-1})^r \frac{1}{l}$$

$$= \sum_{r=0}^{\infty}(-1)^r(\frac{n}{3l}\frac{n-1}{3(l+1)}\cdots\frac{n-r+1}{3(l+r-1)}L^r N^{-r})\frac{1}{l} = \sum_{r=0}^{n}\frac{(-1)^r n(n-1)\ldots(n-r+1)}{3^r l(l+1)\ldots(l+r)} \quad .\Box$$

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Another Formula

While Catalan's formula is elegant, it is not any better than using the recurrence $(*)$ directly, by starting with $a(l+n,0) = 1/(l+n)$ and successively computing $a(l+n-1,1)$, $a(l+n-2,2), \ldots, a(l,n)$, requiring $O(n)$ operations and $O(1)$ storage. The recurrence $(*)$ enables us to quickly find $a(n+1,l-1)$ or $a(n-1,l+1)$, once we know $a(n,l)$, in case the status of one of the children changes. However, neither the recurrence $(*)$, nor Catalan's formula, are efficient if we have already computed $a(l,n)$, and all of a sudden another illegitimate child, that was previously unknown to the family, shows up at the lawyer's office. By using `zeillim` in the package `EKHAD` that accompanies [PWZ], we find the recurrence

$$a(l,n+1) = \frac{1}{l+n+1} + \frac{2(n+1)}{3(l+n+1)} a(l,n) \quad ,$$

that implies the alternative formula:

$$a(l,n) = \frac{n!(2/3)^n}{(n+l)!} \sum_{b=0}^{n} \frac{(l+b-1)!}{b!} (3/2)^b \quad .$$

The Multi-Mistress Case

A typical well-to-do 19th-century French gentleman must have had *plusieurs amantes*. It is very unlikely that he liked them equally well. Let's say that he had m of them, and he wished to leave to every one of the n_i children of Mistress i, x_i of what he would have left them had they been legitimate $(i = 1, \ldots, m)$. Suppose that he also had l legitimate children. What portion of the inheritance should each of these l legitimate children receive? Let's call this quantity $a(l; n_1, \ldots, n_m)$. Then,

$$\left(1 + \sum_{i=1}^{m} \frac{n_i x_i}{l} L N_i^{-1}\right) a(l; n_1, \ldots, n_m) = \frac{1}{l} \quad , \quad \textit{and} \quad \textit{hence}$$

$$a(l; n_1, \ldots, n_m) =$$

$$\left(1 + \sum_{i=1}^{m} \frac{n_i x_i}{l} L N_i^{-1}\right)^{-1} \frac{1}{l} = \sum_{r_1,\ldots,r_m \geq 0}^{\infty} (-1)^{r_1+\ldots+r_m} \frac{(r_1+\ldots+r_m)!}{r_1! \cdots r_m!} \left(\prod_{j=1}^{m} (\frac{x_j n_j}{l} L N_j^{-1})^{r_j}\right) \frac{1}{l}$$

$$= \sum_{r_1,\ldots,r_m \geq 0}^{\infty} (-1)^{r_1+\ldots+r_m} \frac{(r_1+\ldots+r_m)!}{r_1! \cdots r_m!} \left(\prod_{j=1}^{m} x_j^{r_j}\right) \left(\prod_{j=1}^{m} (n_j N_j^{-1})^{r_j}\right) ((1/l)L)^{r_1+\ldots+r_m} \frac{1}{l}$$

$$= \sum_{r_1,\ldots,r_m \geq 0}^{\infty} (-1)^{r_1+\ldots+r_m} \frac{(r_1+\ldots+r_m)!}{r_1! \cdots r_m!} \frac{1}{(l)(l+1)\cdots(l+r_1+\ldots+r_j)} \prod_{j=1}^{m} x_j^{r_j} \prod_{j=1}^{m} \frac{n_j!}{(n_j-r_j)!} \quad .$$

Hence

$$a(l; n_1, \ldots, n_m) = \sum_{r_1=0}^{n_1} \cdots \sum_{r_m=0}^{n_m} \frac{\prod_{j=1}^{m} (-x_j)^{r_j} \binom{n_j}{r_j}}{(1+r_1+\ldots+r_m)\binom{l+r_1+\ldots+r_m}{l-1}} \quad .\Box$$

REFERENCES

[Ca] E.C. Catalan, *Nouv. Ann.*, ser. II, t.2.

[Ch] G. Chrystal, *"ALGEBRA"*, reprinted by Chelsea, New York, NY, 1964.

[PWZ] M. Petkovsek, H.S. Wilf, and D. Zeilberger, *"A=B"*, A.K.Peters, Wellesley, MA, 1996. The accompanying Maple package EKHAD may be downloaded from the author's website.