Automatic Generation of Theorems and Proofs on Enumerating Consecutive-Wilf Classes

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To W from Z (et. al.), a gift for his $\frac{2}{3}|S_5|$*-th birthday*

Preface

This article describes two complementary approaches to enumeration, the *positive* and the *negative*, each with its advantages and disadvantages. Both approaches are amenable to *automation*, and when applied to the currently active subarea, initiated in 2003 by Sergi Elizalde and Marc Noy[EN], of *consecutive pattern-avoidance* in permutations, were successfully pursued by the first two authors Andrew Baxter[B] and Brian Nakamura[N]. This article summarizes their research and in the case of [N] presents an umbral viewpoint to the same approach. The main purpose of this article is to briefly explain the Maple packages, `SERGI` and `ELIZALDE`, developed by AB-DZ and BN-DZ respectively, implementing the algorithms that enable the computer to "do research" by deriving, *all by itself*, functional equations for the generating functions that enable polynomial-time enumeration for any set of patterns. In the case of ELIZALDE (the "negative" approach), these functional equations can be sometimes (automatically!) simplified, and imply "explicit" formulas, that previously were derived by humans using ad-hoc methods. We also get lots of new "explicit" results, beyond the scope of humans, but we have to admit that we still need humans to handle "infinite families" of patterns, but this too, no doubt, will soon be automatable, and we leave this as a challenge to the (human and/or computer) reader.

Consecutive Pattern Avoidance

Inspired by the very active research in pattern-avoidance, pioneered by Herb Wilf, Rodica Simion, Frank Schmidt, Richard Stanley, Don Knuth and others, Sergi Elizalde, in his PhD thesis (written under the direction of Richard Stanley) introduced the study of permutations avoiding *consecutive patterns*.

Recall that an n*-permutation* is a sequence of integers $\pi = \pi_1 \dots \pi_n$ of length n where each integer in $\{1, \dots, n\}$ appears exactly once. It is well-known and very easy to see (today!) that the number of n-permutations is $n! := \prod_{i=1}^{n} i$.

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`http://www.math.rutgers.edu/[~bnaka/, ~baxter/, ~zeilberg/]` . First Version: Jan. 20, 2011. This version: Dec. 13, 2011. We would like to thank the members of the (possibly multiset) $\{referee, VinceVatter\}$ for very helpful remarks.

Accompanied by Maple packages `ELIZALDE` and `SERGI` downloadable from the webpage of this article:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/auto.html` ,
where the reader can find lots of sample input and output. The work of AB and DZ was supported in part by the United States of America National Science Foundation.

The *reduction* of a list of different (integer or real) numbers (or members of any totally ordered set) $[i_1, i_2, \ldots, i_k]$, to be denoted by $R([i_1, i_2, \ldots, i_k])$, is the permutation of $\{1, 2, \ldots, k\}$ that preserves the relative rankings of the entries. In other words, $p_i < p_j$ iff $q_i < q_j$. For example the reduction of $[4, 2, 7, 5]$ is $[2, 1, 4, 3]$ and the reduction of $[\pi, e, \gamma, \phi]$ is $[4, 3, 1, 2]$.

Fixing a pattern $p = [p_1, \ldots, p_k]$, a permutation $\pi = [\pi_1, \ldots, \pi_n]$ *avoids* the consecutive pattern p if for all i, $1 \leq i \leq n - k + 1$, the reduction of the list $[\pi_i, \pi_{i+1}, \ldots, \pi_{i+k-1}]$ is *not* p. More generally a permutation π avoids a set of patterns $\mathcal{P}$ if it avoids each and every pattern $p \in \mathcal{P}$.

The central problem is to answer the question: "Given a pattern or a set of patterns, find a 'formula', or at least an efficient algorithm (in the sense of Wilf[W]), that inputs a positive integer n and outputs the number of permutations of length n that avoid that pattern (or set of patterns)".

Human Research

After the pioneering work of Elizalde and Noy [EN], quite a few people contributed significantly, including Anders Claesson, Toufik Mansour, Sergey Kitaev, Anthony Mendes, Jeff Remmel, and more recently, Vladimir Dotsenko, Anton Khoroshkin and Boris Shapiro. Also recently we witnessed the beautiful resolution of the Warlimont conjecture by Richard Ehrenborg, Sergey Kitaev, and Peter Perry [EKP]. The latter paper also contains extensive references.

Recommended Reading

While the present article tries to be self-contained, the readers would get more out of it if they are familiar with [Z1]. Other applications of the umbral transfer matrix method were given in [EZ][Z2][Z3][Z4].

The Positive Approach vs. the The Negative Approach

We will present two *complementary* approaches to the enumeration of consecutive-Wilf classes, both using the Umbral transfer matrix method. The positive approach works better when you have many patterns, and the negative approach works better when there are only a few, and works best when there is only one pattern to avoid.

Outline of the Positive Approach

Instead of dealing with *avoidance* (the number of permutations that have zero occurrences of the given pattern(s)) we will deal with the more general problem of enumerating the number of permutations that have specified numbers of occurrences of *any* pattern of length k.

Fix a positive integer k, and let $\{t_p : p \in S_k\}$ be $k!$ *commuting indeterminates* (alias variables). Define the *weight* of an n-permutation $\pi = [\pi_1, \ldots, \pi_n]$, to be denoted by $w(\pi)$, by:

$$w([\pi_1, \ldots, \pi_n]) := \prod_{i=1}^{n-k+1} t_{R([\pi_i, \pi_{i+1}, \ldots, \pi_{i+k-1}])} \quad .$$

For example, with $k=3$,

$$w([2,5,1,4,6,3]) := t_{R([2,5,1])}t_{R([5,1,4])}t_{R([1,4,6])}t_{R([4,6,3])} = t_{231}t_{312}t_{123}t_{231} = t_{123}t_{231}^2t_{312} \quad .$$

We are interested in an *efficient* algorithm for computing the sequence of polynomials in $k!$ variables

$$P_n(t_{1\dots k},\dots,t_{k\dots 1}) := \sum_{\pi\in S_n} w(\pi) \quad ,$$

or equivalently, as many terms as desired in the formal power series

$$F_k(\{t_p, p\in S_k\}; z) = \sum_{n=0}^{\infty} P_n z^n \quad .$$

Note that once we have computed the P_n (or F_k), we can answer *any* question about pattern avoidance by specializing the t's. For example to get the number of n-permutations avoiding the single pattern p, of length k, first compute P_n, and then plug-in $t_p=0$ and all the other t's to be 1. If you want the number of n-permutations avoiding the set of patterns $\mathcal{P}$ (all of the same length k), set $t_p=0$ for all $p\in\mathcal{P}$ and the other t's to be 1. As we shall soon see, we will generate *functional equations* for F_k, featuring the $\{t_p\}$ and of course it would be much more efficient to specialize the t_p's to the numerical values already in the functional equations, rather than crank-out the much more complicated $P_n(\{t_p\})$'s and then do the plugging-in.

First let's recall one of the many proofs that the number of n-permutations, let's denote it by $a(n)$, satisfies the recurrence

$$a(n+1) = (n+1)a(n) \quad .$$

Given a typical member of S_n, let's call it $\pi=\pi_1\dots\pi_n$, it can be continued in $n+1$ ways, by deciding on π_{n+1}. If $\pi_{n+1}=i$, then we have to "make room" for the new entry by incrementing by 1 all entries $\geq i$, and then append i. This gives a bijection between $S_n\times[1,n+1]$ and S_{n+1} and taking cardinalities yields the recurrence. Of course $a(0)=1$, and "solving" this recurrence yields $a(n)=n!$. Of course this solving is "cheating", since $n!$ is just shorthand for the solution of this recurrence subject to the initial condition $a(0)=1$, but from now on it is considered "closed form" (just by convention!).

When we do *weighted counting* with respect to the weight w with a given pattern-length k, we have to keep track of the last $k-1$ entries of π:

$$[\pi_{n-k+2}\dots\pi_n] \quad ,$$

and when we append $\pi_{n+1}=i$, the new permutation (let $a'=a$ if $a<i$ and $a'=a+1$ if $a\geq i$)

$$\dots\pi'_{n-k+2}\dots\pi'_n i \quad ,$$

has "gained" a factor of $t_{R[\pi'_{n-k+2}\cdots\pi'_n i]}$ to its weight.

This calls for the finite-state method, alas, the "alphabet" is indefinitely large, so we need the umbral transfer-matrix method.

We introduce $k-1$ "catalytic" variables $x_1, x_2, \ldots, x_{k-1}$, as well as a variable z to keep track of the size of the permutation, and $(k-1)!$ "linear" state variables $A[q]$ for each $q \in S_{k-1}$, to tell us the state that the permutation is in. Define the generalized weight $w'(\pi)$ of a permutation $\pi \in S_n$ to be:

$$w'(\pi) := w(\pi) x_1^{j_1} x_2^{j_2} \ldots x_{k-1}^{j_{k-1}} z^n A[q] \quad ,$$

where $[j_1, \ldots, j_{k-1}]$, $(1 \le j_1 < j_2 < \ldots < j_{k-1} \le n)$ is the *sorted* list of the last $k-1$ entries of π, and q is the reduction of its last $k-1$ entries.

For example, with $k=3$:

$$w'([4,7,1,6,3,5,8,2]) = t_{231}t_{312}t_{132}t_{312}t_{123}t_{231}x_1^2x_2^8z^8A[21] = t_{123}t_{132}t_{231}^2t_{312}^2x_1^2x_2^8z^8A[21] \quad .$$

Let's illustrate the method with $k=3$. There are two states: $[1,2],[2,1]$ corresponding to the cases where the two last entries are j_1j_2 or j_2j_1 respectively (we always assume $j_1 < j_2$) .

Suppose we are in state $[1,2]$, so our permutation looks like

$$\pi = [\ldots, j_1, j_2] \quad ,$$

and $w'(\pi) = w(\pi)x_1^{j_1}x_2^{j_2}z^nA[1,2]$. We want to append i $(1 \le i \le n+1)$ to the end. There are three cases.

Case 1: $1 \le i \le j_1 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_1+1, j_2+1, i] \quad .$$

Its state is $[2,1]$ and $w'(\sigma) = w(\pi)t_{231}x_1^i x_2^{j_2+1}z^{n+1}A[2,1]$.

Case 2: $j_1+1 \le i \le j_2 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_1, j_2+1, i] \quad .$$

Its state is now $[2,1]$ and $w'(\sigma) = w(\pi)t_{132}x_1^i x_2^{j_2+1}z^{n+1}A[2,1]$.

Case 3: $j_2+1 \le i \le n+1 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_1, j_2, i] \quad .$$

Its state is now $[1,2]$ and $w'(\sigma) = w(\pi) t_{123} x_1^{j_2} x_2^{i} z^{n+1} A[1,2]$.

It follows that any *individual* permutation of size n, and state $[1,2]$, gives rise to $n+1$ children, and regarding weight, we have the "umbral evolution" (here W is the fixed part of the weight, that does not change):

$$W x_1^{j_1} x_2^{j_2} z^n A[1,2] \to W t_{231} z A[2,1] \left(\sum_{i=1}^{j_1} x_1^i x_2^{j_2+1} \right) z^n$$

$$+ W t_{132} z A[2,1] \left(\sum_{i=j_1+1}^{j_2} x_1^i x_2^{j_2+1} \right) z^n$$

$$+ W t_{123} z A[1,2] \left(\sum_{i=j_2+1}^{n+1} x_1^{j_2} x_2^{i} \right) z^n \quad .$$

Taking out of the $\sum$-signs whatever we can, we have:

$$W x_1^{j_1} x_2^{j_2} z^n A[1,2] \to W t_{231} z A[2,1] \left(\sum_{i=1}^{j_1} x_1^i \right) x_2^{j_2+1} z^n$$

$$+ W t_{132} z A[2,1] \left(\sum_{i=j_1+1}^{j_2} x_1^i \right) x_2^{j_2+1} z^n$$

$$+ W t_{123} z A[1,2] \left(\sum_{i=j_2+1}^{n+1} x_2^{i} \right) x_1^{j_2} z^n \quad .$$

Now summing up the geometrical series, using the ancient formula:

$$\sum_{i=a}^{b} Z^i = \frac{Z^a - Z^{b+1}}{1-Z} \quad ,$$

we get

$$W x_1^{j_1} x_2^{j_2} z^n A[1,2] \to W t_{231} z A[2,1] \left(\frac{x_1 - x_1^{j_1+1}}{1-x_1} \right) x_2^{j_2+1} z^n$$

$$+ W t_{132} z A[2,1] \left(\frac{x_1^{j_1+1} - x_1^{j_2+1}}{1-x_1} \right) x_2^{j_2+1} z^n$$

$$+ W t_{123} z A[1,2] \left(\frac{x_2^{j_2+1} - x_2^{n+2}}{1-x_2} \right) x_1^{j_2} z^n \quad .$$

This is the same as:

$$W x_1^{j_1} x_2^{j_2} z^n A[1,2] \to W t_{231} z A[2,1] \left(\frac{x_1 x_2^{j_2+1} - x_1^{j_1+1} x_2^{j_2+1}}{1-x_1} \right) z^n$$

$$+Wt_{132}zA[2,1]\left(\frac{x_1^{j_1+1}x_2^{j_2+1}-x_1^{j_2+1}x_2^{j_2+1}}{1-x_1}\right)z^n$$

$$+Wt_{123}zA[1,2]\left(\frac{x_1^{j_2}x_2^{j_2+1}-x_1^{j_2}x_2^{n+2}}{1-x_2}\right)z^n \quad .$$

This is what was called in [Z1], and its many sequels, a "pre-umbra". The above evolution can be expressed for a general *monomial* $M(x_1,x_2,z)$ as:

$$M(x_1,x_2,z)A[1,2] \to t_{231}zA[2,1]\left(\frac{x_1x_2M(1,x_2,z)-x_1x_2M(x_1,x_2,z)}{1-x_1}\right)$$

$$+t_{132}zA[2,1]\left(\frac{x_1x_2M(x_1,x_2,z)-x_1x_2M(1,x_1x_2,z)}{1-x_1}\right)$$

$$+t_{123}zA[1,2]\left(\frac{x_2M(1,x_1x_2,z)-x_2^2M(1,x_1,x_2z)}{1-x_2}\right) \quad .$$

But, by *linearity*, this means that the coefficient of $A[1,2]$ (the weight-enumerator of all permutations of state $[1,2]$) obeys the evolution equation:

$$f_{12}(x_1,x_2,z)A[1,2] \to t_{231}zA[2,1]\left(\frac{x_1x_2f_{12}(1,x_2,z)-x_1x_2f_{12}(x_1,x_2,z)}{1-x_1}\right)$$

$$+t_{132}zA[2,1]\left(\frac{x_1x_2f_{12}(x_1,x_2,z)-x_1x_2f_{12}(1,x_1x_2,z)}{1-x_1}\right)$$

$$+t_{123}zA[1,2]\left(\frac{x_2f_{12}(1,x_1x_2,z)-x_2^2f_{12}(1,x_1,x_2z)}{1-x_2}\right) \quad .$$

Now we have to do it all over for a permutation in state $[2,1]$. Suppose we are in state $[2,1]$, so our permutation looks like

$$\pi = [\ldots, j_2, j_1] \quad ,$$

and $w'(\pi) = w(\pi)x_1^{j_1}x_2^{j_2}z^nA[2,1]$. We want to append i $(1 \le i \le n+1)$ to the end. There are three cases.

Case 1: $1 \le i \le j_1 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_2+1, j_1+1, i] \quad .$$

Its state is $[2,1]$ and $w'(\sigma) = w(\pi)t_{321}x_1^ix_2^{j_1+1}z^{n+1}A[2,1]$.

Case 2: $j_1+1 \le i \le j_2 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_2+1, j_1, i] \quad .$$

Its state is now $[1,2]$ and

$w'(\sigma) = w(\pi) t_{312} x_1^{j_1} x_2^{i} z^{n+1} A[1,2]$.

Case 3: $j_2 + 1 \leq i \leq n+1 \quad .$

The new permutation, let's call it σ, looks like

$$\sigma = [\ldots j_2, j_1, i] \quad .$$

Its state is $[1,2]$ and $w'(\sigma) = w(\pi) t_{213} x_1^{j_1} x_2^{i} z^{n+1} A[1,2]$.

It follows that any *individual* permutation of size n, and state $[2,1]$, gives rise to $n+1$ children, and regarding weight, we have the "umbral evolution" (here W is the fixed part of the weight, that does not change):

$$W x_1^{j_1} x_2^{j_2} z^n A[2,1] \rightarrow W t_{321} z A[2,1] \left(\sum_{i=1}^{j_1} x_1^{i} x_2^{j_1+1} \right) z^n$$

$$+ W t_{312} z A[1,2] \left(\sum_{i=j_1+1}^{j_2} x_1^{j_1} x_2^{i} \right) z^n$$

$$+ W t_{213} z A[1,2] \left(\sum_{i=j_2+1}^{n+1} x_1^{j_1} x_2^{i} \right) z^n \quad .$$

Taking out of the $\sum$-signs whatever we can, we have:

$$W x_1^{j_1} x_2^{j_2} z^n A[2,1] \rightarrow W t_{321} z A[2,1] \left(\sum_{i=1}^{j_1} x_1^{i} \right) x_2^{j_1+1} z^n$$

$$+ W t_{312} z A[1,2] \left(\sum_{i=j_1+1}^{j_2} x_2^{i} \right) x_1^{j_1} z^n$$

$$+ W t_{213} z A[1,2] \left(\sum_{i=j_2+1}^{n+1} x_2^{i} \right) x_1^{j_1} z^n \quad .$$

Now summing up the geometrical series, using the ancient formula:

$$\sum_{i=a}^{b} Z^i = \frac{Z^a - Z^{b+1}}{1-Z} \quad ,$$

we get

$$W x_1^{j_1} x_2^{j_2} z^n A[2,1] \rightarrow W t_{321} z A[2,1] \left(\frac{x_1 - x_1^{j_1+1}}{1-x_1} \right) x_2^{j_1+1} z^n$$

$$+Wt_{312}zA[1,2]\left(\frac{x_2^{j_1+1}-x_2^{j_2+1}}{1-x_2}\right)x_1^{j_1}z^n$$

$$+Wt_{213}zA[1,2]\left(\frac{x_2^{j_2+1}-x_2^{n+2}}{1-x_2}\right)x_1^{j_1}z^n \quad .$$

This is the same as:

$$Wx_1^{j_1}x_2^{j_2}z^nA[2,1]\to Wt_{321}zA[2,1]\left(\frac{x_1x_2^{j_1+1}-x_1^{j_1+1}x_2^{j_1+1}}{1-x_1}\right)z^n$$

$$+Wt_{312}zA[1,2]\left(\frac{x_1^{j_1}x_2^{j_1+1}-x_1^{j_1}x_2^{j_2+1}}{1-x_2}\right)z^n$$

$$+Wt_{213}zA[1,2]\left(\frac{x_1^{j_1}x_2^{j_2+1}-x_1^{j_1}x_2^{n+2}}{1-x_2}\right)z^n \quad .$$

The above evolution can be expressed for a general *monomial* $M(x_1,x_2,z)$ as:

$$M(x_1,x_2,z)A[2,1]\to t_{321}zA[2,1]\left(\frac{x_1x_2M(x_2,1,z)-x_1x_2M(x_1x_2,1,z)}{1-x_1}\right)$$

$$+t_{312}zA[1,2]\left(\frac{x_2M(x_1x_2,1,z)-x_2M(x_1,x_2,z)}{1-x_2}\right)$$

$$+t_{213}zA[1,2]\left(\frac{x_2M(x_1,x_2,z)-x_2^2M(x_1,1,x_2z)}{1-x_2}\right) \quad .$$

But, by *linearity*, this means that the coefficient of $A[2,1]$ (the weight-enumerator of all permutations of state $[2,1]$) obeys the evolution equation:

$$f_{21}(x_1,x_2,z)A[2,1]\to t_{321}zA[2,1]\left(\frac{x_1x_2f_{21}(x_2,1,z)-x_1x_2f_{21}(x_1x_2,1,z)}{1-x_1}\right)$$

$$+t_{312}zA[1,2]\left(\frac{x_2f_{21}(x_1x_2,1,z)-x_2f_{21}(x_1,x_2,z)}{1-x_2}\right)$$

$$+t_{213}zA[1,2]\left(\frac{x_2f_{21}(x_1,x_2,z)-x_2^2f_{21}(x_1,1,x_2z)}{1-x_2}\right) \quad .$$

Combining we have the "evolution":

$$f_{12}(x_1,x_2,z)A[1,2]+f_{21}(x_1,x_2,z)A[2,1]\to$$

$$t_{231}zA[2,1]\left(\frac{x_1x_2f_{12}(1,x_2,z)-x_1x_2f_{12}(x_1,x_2,z)}{1-x_1}\right)$$

$$+t_{132}zA[2,1]\left(\frac{x_1x_2f_{12}(x_1,x_2,z)-x_1x_2f_{12}(1,x_1x_2,z)}{1-x_1}\right)$$

$$+t_{123}zA[1,2]\left(\frac{x_2f_{12}(1,x_1x_2,z)-x_2^2f_{12}(1,x_1,x_2z)}{1-x_2}\right) \quad .$$

$$+t_{321}zA[2,1]\left(\frac{x_1x_2f_{21}(x_2,1,z)-x_1x_2f_{21}(x_1x_2,1,z)}{1-x_1}\right)$$

$$+t_{312}zA[1,2]\left(\frac{x_2f_{21}(x_1x_2,1,z)-x_2f_{21}(x_1,x_2,z)}{1-x_2}\right)$$

$$+t_{213}zA[1,2]\left(\frac{x_2f_{21}(x_1,x_2,z)-x_2^2f_{21}(x_1,1,x_2z)}{1-x_2}\right) \quad .$$

Now the "evolved" (new) $f_{12}(x_1,x_2,z)$ and $f_{21}(x_1,x_2,z)$ are the coefficients of $A[1,2]$ and $A[2,1]$ respectively, and since the *initial weight* of both of them is $x_1x_2^2z^2$, we have the established the following system of functional equations:

$$f_{12}(x_1,x_2,z)=x_1x_2^2z^2$$

$$+t_{123}z\left(\frac{x_2f_{12}(1,x_1x_2,z)-x_2^2f_{12}(1,x_1,x_2z)}{1-x_2}\right)$$

$$+t_{312}z\left(\frac{x_2f_{21}(x_1x_2,1,z)-x_2f_{21}(x_1,x_2,z)}{1-x_2}\right)$$

$$+t_{213}z\left(\frac{x_2f_{21}(x_1,x_2,z)-x_2^2f_{21}(x_1,1,x_2z)}{1-x_2}\right) \quad ,$$

and

$$f_{21}(x_1,x_2,z)=x_1x_2^2z^2$$

$$+t_{231}z\left(\frac{x_1x_2f_{12}(1,x_2,z)-x_1x_2f_{12}(x_1,x_2,z)}{1-x_1}\right)$$

$$+t_{132}z\left(\frac{x_1x_2f_{12}(x_1,x_2,z)-x_1x_2f_{12}(1,x_1x_2,z)}{1-x_1}\right)$$

$$+t_{321}z\left(\frac{x_1x_2f_{21}(x_2,1,z)-x_1x_2f_{21}(x_1x_2,1,z)}{1-x_1}\right) \quad .$$

Let the computer do it!

All the above was only done for *pedagogical* reasons. The computer can do it all automatically, much faster and more reliably. Now if we want to find functional equations for the number of permutations avoiding a given set of consecutive patterns $\mathcal{P}$, all we have to do is plug-in $t_p=0$ for $p\in\mathcal{P}$ and $t_p=1$ for $p\notin\mathcal{P}$. This gives a polynomial-time algorithm for computing any desired number of terms. This is all done automatically in the Maple package `SERGI`. See the webpage of this article for lots of sample input and output.

Above we assumed that the members of the set P are all of the same length, k. Of course more general scenarios can be reduced to this case, where k would be the largest length that shows up in P. Note that with this approach we end up with a set of $(k-1)!$ functional equations in the $(k-1)!$ "functions" (or rather formal power series) f_p.

The Negative Approach

Suppose that we want to compute quickly the first 100 terms (or whatever) of the sequence enumerating n-permutations avoiding the pattern $[1, 2, \ldots, 20]$. As we have already noted, using the "positive" approach, we have to set-up a *system* of functional equations with 19! equations and 19! unknowns. While the algorithm is still *polynomial* in n (and would give a "Wilfian" answer), it is not very practical! (This is yet another illustration why the ruling paradigm in theoretical computer science, of equating "polynomial time" with "fast" is (sometimes) absurd).

This is analogous to computing words in a *finite* alphabet, say of a letters, avoiding a given word (or words) as *factors* (consecutive subwords). If the word-to-avoid has length k, then the naive transfer-matrix method would require setting up a system of a^{k-1} equations and a^{k-1} unknowns. The elegant and powerful *Goulden-Jackson method* [GJ1][GJ2], beautifully exposited and extended in [NZ], and even further extended in [KY], enables one to do it by solving one equation in one unknown. We assume that the reader is familiar with it, and briefly describe the analog for the present problem, where the alphabet is "infinite". This is also the approach pursued in the beautiful human-generated papers [DK] and [KS]. We repeat that the *focus* and *novelty* in the present work is in *automating* enumeration, and the current topic of consecutive pattern-avoidance is used as a *case-study.*

First, some generalities! For ease of exposition, let's focus on a single pattern p (the case of several patterns is analogous, see [DK]).

Using the inclusion-exclusion "negative" philosophy for counting, fix a pattern p. For any n-permutation, let $Patt_p(\pi)$ be the set of occurrences of the pattern p in π. For example

$$Patt_{123}(179234568) = \{179, 234, 345, 456, 568\} \quad ,$$

$$Patt_{231}(179234568) = \{792\} \quad ,$$

$$Patt_{312}(179234568) = \{923\} \quad ,$$

$$Patt_{132}(179234568) = Patt_{213}(179234568) = Patt_{321}(179234568) = \emptyset \quad .$$

Consider the much larger set of pairs

$$\{[\pi, S] | \ \pi \in S_n \quad , \ S \subset Patt_p(\pi)\},$$

and define

$$weight_p[\pi, S] := (t-1)^{|S|} \quad ,$$

where $|S|$ is the number of elements of S. For example,

$$weight_{123}[179234568, \{234, 568\}] = (t-1)^2 \quad ,$$

$$weight_{123}[179234568, \{179\}] = (t-1)^1 = t-1 \quad ,$$

$$weight_{123}[179234568, \emptyset] = (t-1)^0 = 1 \quad .$$

Fix a (consecutive) pattern p of length k, and consider the weight-enumerator of all n-permutations according to the weight

$$w(\pi) := t^{\#occurrences\ of\ pattern\ p\ in\ \pi} \quad ,$$

let's call it $P_n(t)$. So:

$$P_n(t) := \sum_{\pi \in S_n} t^{|Patt_p(\pi)|} \quad .$$

Now we need the *crucial*, extremely deep, fact:

$$t = (t-1) + 1 \quad ,$$

and its corollary (for any finite set S):

$$t^{|S|} = ((t-1)+1)^{|S|} = \prod_{s \in S} ((t-1)+1) = \sum_{T \subset S} (t-1)^{|T|} \quad .$$

Putting this into the definition of $P_n(t)$, we get:

$$P_n(t) := \sum_{\pi \in S_n} t^{|Patt_p(\pi)|} = \sum_{\pi \in S_n} \sum_{T \subset Patt_p(\pi)} (t-1)^{|T|} \quad .$$

This is the weight-enumerator (according to a different weight, namely $(t-1)^{|T|}$) of a much larger set, namely the set of *pairs*, (π, T), where T is a subset of $Patt_p(\pi)$. Surprisingly, this is much easier to handle!

Consider a typical such "creature" (π, T). There are two cases

Case I: The last entry of π, π_n does not belong to any of the members of T, in which case chopping it off produces a shorter such creature, in the set $\{1, 2, \dots, n\} \backslash \{\pi_n\}$, and reducing both π and T to $\{1, \dots, n-1\}$ yields a typical member of size $n-1$. Since there are n choices for π_n, the weight-enumerator of creatures of this type (where the last entry does not belong to any member of T) is $nP_{n-1}(t)$.

Case II: Let's order the members of T by their first (or last) index:

$$[s_1, s_2, \dots, s_p] \quad ,$$

where the last entry of π, π_n, belongs to s_p. If s_p and s_{p-1} are disjoint, the ending cluster is simply $[s_p]$. Otherwise s_p intersects s_{p-1}. If s_{p-1} and s_{p-2} are disjoint, then the ending cluster is $[s_{p-1}, s_p]$. More generally, the ending cluster of the pair $[\pi, [s_1, \dots, s_p]]$ is the unique list $[s_i, \dots, s_p]$ that has the property that s_i intersects s_{i+1}, s_{i+1} intersects s_{i+2}, ..., s_{p-1} intersects s_p, but s_{i-1} does not intersect s_i. It is possible that the ending cluster of $[\pi, T]$ is the whole T.

Let's give an example: with the pattern 123. The ending cluster of the pair:

$$[157423689, [157, 236, 368, 689]]$$

is $[236, 368, 689]$ since 236 overlaps with 368 (in two entries) and 368 overlaps with 689 (also in two entries), while 157 is disjoint from 236.

Now if you remove the ending cluster of T from T and remove the entries participating in the cluster from π, you get a shorter creature $[\pi', T']$ where π' is the permutation with all the entries in the ending cluster removed, and T' is what remains of T after we removed that cluster. In the above example, we have

$$[\pi', T'] = [1574, [157]] \quad .$$

Suppose that the length of π' is r.

Let $C_n(t)$ be the weight-enumerator, according to the weight $(t-1)^{|T|}$, of canonical clusters of length n, i.e., those whose set of entries is $\{1, \dots, n\}$. Then in Case II we have to choose a subset of $\{1, \dots, n\}$ of cardinality $n-r$ to be the set of entries of $[\pi', T']$ and then choose a creature of size $n-r$ and a cluster of size r. Combining Case I and Case II, we have , $P_0(t) = 1$, and for $n \geq 1$:

$$P_n(t) = nP_{n-1}(t) + \sum_{r=2}^{n} \binom{n}{r} P_{n-r}(t) C_r(t) \quad .$$

Now it is time to consider the *exponential generating function*

$$F(z,t) := \sum_{n=0}^{\infty} \frac{P_n(t)}{n!} z^n \quad .$$

We have

$$F(z,t) := 1 + \sum_{n=1}^{\infty} \frac{P_n(t)}{n!} z^n = 1 + \sum_{n=1}^{\infty} \frac{nP_{n-1}(t)}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{r=2}^{n} \binom{n}{r} P_{n-r}(t) C_r(t) \right) z^n$$

$$= 1 + z \sum_{n=1}^{\infty} \frac{P_{n-1}(t)}{(n-1)!} z^{n-1} + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{r=2}^{n} \frac{n!}{r!(n-r)!} P_{n-r}(t) C_r(t) \right) z^n$$

$$= 1 + z \sum_{n=0}^{\infty} \frac{P_n(t)}{n!} z^n + \sum_{n=0}^{\infty} \left(\sum_{r=2}^{n} \frac{1}{r!(n-r)!} P_{n-r}(t) C_r(t) \right) z^n$$

$$= 1 + zF(z,t) + \sum_{n=0}^{\infty} \left(\sum_{r=2}^{n} \frac{P_{n-r}(t)}{(n-r)!} \frac{C_r(t)}{r!} \right) z^n$$

$$= 1 + zF(z,t) + \left(\sum_{n-r=0}^{\infty} \frac{P_{n-r}(t)}{(n-r)!} z^{n-r} \right) \left(\sum_{r=0}^{\infty} \frac{C_r(t)}{r!} z^r \right) \quad ,$$

since $C_0(t) = 0, C_1(t) = 0$, and this equals

$$= 1 + zF(z,t) + F(z,t)G(z,t) \quad ,$$

where $G(z,t)$ is the exponential generating function of $C_n(t)$:

$$G(z,t) := \sum_{n=0}^{\infty} \frac{C_n(t)}{n!} z^n \quad .$$

It follows that

$$F(z,t) = 1 + zF(z,t) + F(z,t)G(z,t) \quad ,$$

leading to

$$F(z,t) = \frac{1}{1 - z - G(z,t)} \quad .$$

So if we had a quick way to compute the sequence $C_n(t)$, we would have a quick way to compute the first *whatever* coefficients (in z) of $F(z,t)$ (i.e., as many $P_n(t)$ as desired).

A Fast Way to compute $C_n(t)$

For the sake of pedagogy let the fixed pattern be 1324. Consider a typical cluster

$$[13254768, [1325, 2547, 4768]] \quad .$$

If we remove the last atom of the cluster, we get the cluster

$$[132547, [1325, 2547]] \quad ,$$

of the set $\{1,2,3,4,5,7\}$. Its canonical form, reduced to the set $\{1,2,3,4,5,6\}$, is:

$$[132546, [1325, 2546]] \quad .$$

Because of the "Markovian property" (chopping the last atom of the clusters and reducing yields a shorter cluster), we can build-up such a cluster, and in order to know how to add another atom, all we need to know is the current last atom. If the pattern is of length k (in this example, $k=4$), we need only to keep track of the last k entries. Let the sorted list (from small to large) be $i_1 < \ldots < i_k$, so the last atom of the cluster (with r atoms) is $s_r = [i_{p_1}, \ldots, i_{p_k}]$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is some increasing sequence of k integers between 1 and n. We introduce k *catalytic* variables $x_1, \ldots, x_k$, and define

$$Weight([s_1, \ldots, s_{r-1}, [i_{p_1}, \ldots, i_{p_k}]]) := z^n (t-1)^r x_1^{i_1} \cdots x_k^{i_k} \quad .$$

Going back to the 1324 example, if we currently have a cluster with r atoms, whose last atom is $[i_1, i_3, i_2, i_4]$, how can we add another atom? Let's call it $[j_1, j_3, j_2, j_4]$. This new atom can overlap with the former one in two possibilities.

(a) If the overlap is of length 2:

$$j_1 = i_2 \quad j_3 = i_4 \quad ,$$

but because of the "reduction" (making room for the new entries) it is really

$$j_1 = i_2 \quad j_3 = i_4 + 1 \quad ,$$

(and j_2 and j_4 can be what they wish as long as $i_2 < j_2 < i_4 + 1 < j_4 \le n$).

(b) If the overlap is of length 1:

$$j_1 = i_4$$

(and j_2, j_3, j_4 can be what they wish, provided that $i_4 < j_2 < j_3 < j_4 \le n$).

Hence we have the "umbral-evolution":

$$z^n(t-1)^{r-1}x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4} \to z^{n+2}(t-1)^r \sum_{1 \le j_1 = i_2 < j_2 < j_3 = i_4+1 < j_4 \le n} x_1^{j_1}x_2^{j_2}x_3^{j_3}x_4^{j_4}$$

$$+z^{n+3}(t-1)^r \sum_{1 \le j_1 = i_4 < j_2 < j_3 < j_4 \le n} x_1^{j_1}x_2^{j_2}x_3^{j_3}x_4^{j_4} \quad .$$

These two iterated geometrical sums can be summed exactly, and from this "pre-umbra" the computer can deduce (automatically!) the umbral operator, yielding a functional equation for the **ordinary** generating function

$$\mathcal{C}(t, z; x_1, \dots, x_k) = \sum_{n=0}^{\infty} C_n(t; x_1, \dots, x_k) z^n \quad ,$$

of the form

$$\mathcal{C}(t, z; x_1, \dots, x_k) = (t-1)z^k x_1 x_2^2 \dots x_k^k + \sum_{\alpha} R_\alpha(x_1, \dots, x_k; t, z)\mathcal{C}(t, z; M_1^\alpha, \dots, M_k^\alpha) \quad ,$$

where $\{\alpha\}$ is a finite index set, $M_1^\alpha, \dots, M_k^\alpha$ are specific monomials in $x_1, \dots, x_k, z$, derived by the algorithm, and R_α are certain rational functions of their arguments, also derived by the algorithm.

Once again, the novelty here is that everything (except for the initial Maple programming) is done *automatically* by the computer. It is the computer doing combinatorial research all on its own!

Post-Processing the Functional Equation

At the end of the day we are only interested in $\mathcal{C}(t, z; 1, \dots, 1)$. Alas, plugging in $x_1 = 1, x_2 = 1, \dots, x_k = 1$ would give lots of $0/0$. Taking the limits, and using L'Hôpital, is an option, but then we get a differential equation that would introduce differentiations with respect to the catalytic variables, and we would not gain anything.

But it so happens, in many cases, that the functional operator preserves some of the exponents of the $x_i's$. For example for the pattern 321 the last three entries are always $[3, 2, 1]$, and one can do a *change of dependent variable*:

$$\mathcal{C}(t, z; x_1, \dots, x_3) = x_1 x_2^2 x_3^3 g(z; t) \quad ,$$

and *now* plugging in $x_1 = 1, x_2 = 1, x_3 = 1$ is harmless, and one gets a much simpler functional equation with *no* catalytic variables, that turns out to be (according to S.B. Ekhad) the simple algebraic equation

$$g(z,t) = -(t-1)z^2 - (t-1)(z+z^2)g(z,t) \quad ,$$

that in this case can be solved in closed-form (reproducing a result that goes back to [EN]). Other times (like the pattern 231), we only get rid of some of the catalytic variables. Putting

$$\mathcal{C}(t, z; x_1, \ldots, x_3) = x_1 x_2^2 g(x_3, z; t) \quad ,$$

(and then plugging in $x_1 = 1, x_2 = 1$) gives a much simplified functional equation, and now taking the limit $x_3 \to 1$ and using L'Hôpital (that Maple does all by itself) one gets a pure differential equation for $g(1, z; t)$, in z, that sometimes can be even solved in closed form (automatically by Maple). But from the point of view of efficient enumeration, it is just as well to leave it at that.

Any pattern p is trivially equivalent to (up to) three other patterns (its reverse, its complement, and the reverse-of-the-complement, some of which may coincide). It turns out that out of these (up to) four options, there is one that is easiest to handle, and the computer finds this one, by finding which ones gives the simplest functional (or, if in luck, differential or algebraic) equation, and goes on to handle only this representative.

The Maple package ELIZALDE

All of this is implemented in the Maple package `ELIZALDE`, that automatically produces *theorems* and *proofs*. Lots of sample output (including computer-generated theorems and *proofs*) can be found on the webpage of this article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/auto.html` .

In particular, to see all theorems and *proofs* for patterns of lengths 3 through 5 go to (respectively):

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oEP3_200` ,

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oEP4_60` ,

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oEP5_40` .

If the proofs bore you, and by now you believe Shalosh B. Ekhad, and you only want to see the statements of the *theorems*, for lengths 3 through 6 go to (respectively):

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oET3_200` ,

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oET4_60` ,

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oET5_40` .

`http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oET6_30` .

Humans, with their short attention spans, would probably soon get tired of even the statements of most of the theorems of this last file (for patterns of length 6).

In addition to "symbol crunching" this package does quite a lot of "number crunching" (of course using the former). To see the "hit parade", ranked by size, together with the conjectured asymptotic growth for single consecutive-pattern avoidance of lengths between 3 and 6, see, respectively, the output files:

http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oE3_200 ,

http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oE4_60 ,

http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oE5_40 ,

http://www.math.rutgers.edu/ zeilberg/tokhniot/sergi/oE6_30 .

Enjoy!

References

[B] Andrew Baxter, *"Algorithms For Permutation Statistics"*, *PhD dissertation*, Rutgers University, May 2011. Available from
`http://www.math.rutgers.edu/~zeilberg/Theses/AndrewBaxterThesis.pdf`.

[DK] Vladimir Dotsenko and Anton Khoroshkin, *Anick-type resolutions and consecutive pattern-avoidance*, arXiv:1002.2761v1[Math.CO] .

[EKP] Richard Ehrenborg, Sergey Kitaev, and Peter Perry, *A Spectral Approach to Consecutive Pattern-Avoiding Permutations*, arXiv: 1009.2119v1 [math.CO] 10 Sep 2010 .

[EN] Sergi Elizalde and Marc Noy, *Consecutive patterns in permutations*, Advances in Applied Mathematics **30** (2003), 110-125.

[EZ] Shalosh B. Ekhad, and D. Zeilberger, *Using Rota's Umbral Calculus to Enumerate Stanley's P-Partitions*, Advances in Applied Mathematics **41** (2008), 206-217.

[GJ1] Ian Goulden and David M. Jackson, *An inversion theorem for cluster decompositions of sequences with distinguished subsequences*, J. London Math. Soc.(2)**20** (1979), 567-576.

[GJ2] Ian Goulden and David M. Jackson, *"Combinatorial Enumeration"*, John Wiley, 1983, New York.

[KS] Anton Khoroshkin and Boris Shapiro, *Using homological duality in consecutive pattern avoidance*, arXiv:1009.5308v1 [math.CO].

[KY] Elizabeth J. Kupin and Debbie S. Yuster, *Generalizations of the Goulden-Jackson Cluster*

Method, J. Difference Eq. Appl. **16** (2010), 1563-5120. arXiv:0810.5113v1[math.CO] .

[N] Brian Nakamura, *Computational Approaches to Consecutive Pattern Avoidance in Permutations*, submitted, available from `http://arxiv.org/abs/1102.2480` .

[NZ] John Noonan and Doron Zeilberger, *The Goulden-Jackson Cluster Method: Extensions, Applications, and Implementations*, J. Difference Eq. Appl. **5** (1999), 355-377.

[W] Herbert S. Wilf, *What is an answer*, Amer. Math. Monthly **89** (1982), 289-292.

[Z1] Doron Zeilberger, *The Umbral Transfer-Matrix Method I. Foundations*, J. Comb. Theory, Ser. A **91** (2000), 451-463 .

[Z2] Doron Zeilberger, *The Umbral Transfer-Matrix Method. III. Counting Animals*, New York Journal of Mathematics **7**(2001), 223-231 .

[Z3] Doron Zeilberger, *The Umbral Transfer-Matrix Method V. The Goulden-Jackson Cluster Method for Infinitely Many Mistakes*, Integers **2** (2002), A5.

[Z4] Doron Zeilberger, *In How Many Ways Can You Reassemble Several Russian Dolls?*, Personal Journal of S.B. Ekhad and D. Zeilberger, `http://www.math.rutgers.edu/~zeilberg/pj.html`, Sept. 16, 2009.