A Proof in the Style of George Andrews
(and G. H. Hardy, and Unfortunately MANY other, otherwise very smart, people)
That $\sum_{i=0}^{n} 1 = n+1$

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Theorem: Let n be a non-negative integer, then we have the following identity

$$\sum_{i=0}^{n} 1 = n+1 \quad .$$

Proof: Recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} 1 \cdot z^n \quad ,$$

valid for all complex numbers z such that $|z| < 1$. Squaring both sides, we get

$$\left(\frac{1}{1-z}\right)^2 = \left(\sum_{n=0}^{\infty} 1 \cdot z^n\right)^2 \quad .$$

Hence

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} 1 \cdot 1) z^n \quad = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} 1) z^n \quad , \tag{1}$$

also valid for $|z| < 1$. Since the function on the left side is analytic in $|z| < 1$, it has a Taylor expansion at $z = 0$. By induction on n, (and using the chain rule) the reader can verify that

$$\frac{d^n}{dz^n} \frac{1}{(1-z)^2} = \frac{(n+1)!}{(1-z)^{n+2}} \quad ,$$

also valid for $|z| < 1$. Hence

$$\frac{d^n}{dz^n} \frac{1}{(1-z)^2} \Big|_{z=0} = (n+1)! \quad .$$

By Taylor's theorem

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} z^n = \sum_{n=0}^{\infty} (n+1) z^n \quad , \tag{2}$$

valid for $|z| < 1$. Since the right sides of Eqs. (1) and Eqs. (2) are both analytic functions that are equal to each other in an open neighborhood of $z = 0$ (that happens to be the open disk $|z| < 1$), we can equate the coefficients of z^n, getting the desired identity. $\Box$.

Comments. In case you didn't notice, this is a **PARODY** of the very bad habit, (that like so many bad habits seems so hard to get rid of) of using analytic proofs where they are not needed.

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While the above proof is technically correct, it is *morally* **worse** than wrong, since it constitutes a **category mistake**, that according to wikipedia is when *"things of one kind are presented as if they belonged to another"*. The statement of the "theorem" is entirely elementary, dealing with integers, and while I am sure that the reader can find a "direct" proof (hint: $1+1+\ldots+1$ (a times) equals a), the above "proof" can be recast in the *purely algebraic* (and much more elementary!) (fully rigorous [in fact far more rigorous]) *algebra of formal power series*, and ditto for much "deeper" identities, including the Rogers-Ramanujan ones!

Of course George Andrews knows it very well, and explicitly states a "disclaimer" on p. 3 of his classic book *Theory of Partitions*, that "for many problems it suffices to consider the generating functions as *formal power series*", and then the manipulations become "almost trivial". But because that for *some* applications (to asymptotics) one seems to need analytic convergence, he made the very **bad decision** to consider *all* generating functions as belonging to *analytic functions*. Indeed a serious *category mistake*.

I was lead to this parody while reading a recent (otherwise beautiful!) article by S. Ole Warnaar (Notices of the Amer. Math. Soc. #60(1) (Jan. 2013), pp. 18-22) about Ramanujan's ${}_1\psi_1$ Summation. In that article, Warnaar repeated the conventional statement of this celebrated identity (most probably Hardy's *bowdlerized* version, Ramanujan was a formalist at heart!) viewing both sides as *analytic functions* "valid for $|b/a| < |z| < 1$", rather than viewing them, much more naturally, as "*bilateral formal power series* in z with coefficients that are rational functions of b", and then goes on to give Mourad Ismail's original *proof-from-the-book*, rather than my own elementary adaptation of Ismail's gorgeous proof, sketched in

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/cfinite.pdf` .

Warnaar then goes on to describe some amazing *almost trivial* consequences, for example he deduced, by a clever specialization, Jacobi's famous theorem about the number of ways to represent a positive integer as as sum of four squares, leaving the impression that analysis is required to prove this beautiful but purely elementary statement, thereby propagating Andrews' (and Hardy's and Ismail's and Askey's and ...) pernicious *category mistake*.

I am sure that in many other parts of both pure and applied mathematics, where analysis seems indispensable, it is a mere *red herring* due to our human limitations. Let's work together to free us from the annoying shackles of *continuous* analysis, and replace it by the much healthier and prettier *discrete* analysis. Amen.