Some Remarks on a recent article by J. -P. Allouche

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In a recent article [A], the author tries very hard to "explain" a puzzling remark made by Otto G. Ruehr in his solution [R] to American Mathematical Monthly elementary problem **E2765** (proposed in 1979 by Naoki Kimura).

Frankly, we don't see the point of going to such great lengths to prove that routinely-provable identity A implies routinely-provable identity B. It is much faster to prove them **both** from scratch. As we will soon see, Ruehr's remark was most probably a *non-sequitur*, and it is very unlikely that it followed [A]'s *exegesis*.

The original problem was to prove that for every continuous function f the following identity holds

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} f(3x^2-2x^3)\,dx \;=\; 2\int_0^1 f(3x^2-2x^3)\,dx \quad .$$

After giving his change-of-variable proof, Ruehr makes the following remark.

"An interesting alternative approach would be to use the Weierstrass Approximation Theorem to reduce that problem to that of a polynomial f, and in turn, by linearity to that of establishing the given equation for $f(z)=z^n$."

In other words one had to prove, that for every non-negative integer n,

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} (3x^2-2x^3)^n\,dx \;=\; \int_0^1 2\,(3x^2-2x^3)^n\,dx \quad . \qquad (Ruehr)$$

While this was non-trivial back in 1980, it is routinely provable today, using the *Almkvist-Zeilberger* algorithm [AZ], implemented in the Maple package `EKHAD.txt` [Z]. Just type:

```
AZdI((3*x**2-2*x**3)**n,x,n,N,-1/2,3/2)[1];   AZdI(2*(3*x**2-2*x**3)**n,x,n,N,0,1)[1];
```

for the left and right sides, respectively, and in a split second you would get the **same** output:

```
[9 (n + 1) (2 n + 1) - 2 (3 n + 4) (3 n + 2) N, 2]    ,
```

which means that **both** the left side and the right side satisfy the first-order inhomogeneous linear recurrence equation with polynomial coefficients

$$9(n+1)(2n+1)f(n) \,-\, 2(3n+4)(3n+2)f(n+1) \;=\; 2 \quad .$$

Since $L(0)=R(0)=2$, this immediately implies $(Ruehr)$, without any need for a clever change of variable.

If you take Ruehr's suggestion literally, then you would get a certain binomial coefficients identity that is easily provable by the Zeilberger algorithm, supplying yet another routine proof. This summation identity, that the reader can easily derive by using the binomial theorem on $(3-2x)^n$ and integrating term-by-term (as suggested in [R]) has **nothing whatsoever to do** with the following two binomial coefficients identities that Ruehr claims are 'equivalent'. We would never know what Ruehr had in mind (he probably got mixed up with another problem), and frankly we don't really care. At any rate, we are sure that it is not via Allouche's extremely circuitous route.

These 'equivalent' (per Ruehr) identities are

$$\sum_{j=0}^{2n} (-4)^j \binom{3n+1}{n+j+1} = \sum_{j=0}^{n} 2^j \binom{3n+1}{n-j} \quad ,$$

$$\sum_{j=0}^{2n} (-3)^j \binom{3n-j}{n} = \sum_{j=0}^{n} 3^j \binom{3n-j}{2n} \quad .$$

As already pointed out in [MTWZ] these two new identities (that, in spite of [A], are completely unrelated to the original integral identity (*Ruehr*)) are also **routinely provable** with Maple. In fact it is also strange that they are listed as two different identities. All four sums happen to be identical.

Go into Maple, and type

```
Z:=SumTools[Hypergeometric][ZeilbergerRecurrence];
```

,

then type

```
Z(3**j*binomial(3*n-j,2*n),n,j,f,0..n);   Z((-3)**j*binomial(3*n-j,n),n,j,f,0..2*n);

Z(2**j*binomial(3*n+1,n-j),n,j,f,0..n); Z((-4)**j*binomial(3*n+1,n+j+1),n,j,f,0..2*n);
```

and find out, in one nano-second, that all four sums satisfy the **same** linear recurrence, namely

$$-27\, f\,(n) + 4\, f\,(n+1) \;=\; -3\, \frac{(3n+1)!}{(2n+1)!(n+1)!}$$

As also pointed out in [MTWZ] these four sums are all equal to OEIS sequence `A6256` [S] whose definition is yet another binomial coefficient sum

$$\sum_{k=0}^{n} \binom{3k}{k} \binom{3n-3k}{n-k} \quad .$$

Typing

```
Z(binomial(3*k,k)*binomial(3*n-3*k,n-k),n,k,f,0..n);
```

gives that it satisfies a second-order **homogeneous** linear recurrence equation with polynomial coefficients

$$-81\,(3\,n+2)\,(3\,n+4)\,f\,(n)+\left(216\,n^2+594\,n+420\right)\,f\,(n+1)-8\,(2\,n+3)\,(n+2)\,f\,(n+2)\;=\;0\quad,$$

that is routinely equivalent to the above first-order inhomogeneous recurrence for the Ruehr sums.

Conclusion

In the conclusion to [A], the author states

"The literature about sums involving binomial coefficients is huge."

He should have added,

"... and mostly obsolete[1] *(thanks to the Wilf-Zeilberger [PWZ] algorithmic proof theory, implemented in Maple, Mathematica and other systems)".*

Both the integral formula with $f(z)=z^n$ and the two binomial coefficients identities, are nowadays routinely provable, and it has very little *mathematical* interest to do *exegesis* of what Ruehr had in mind. Of course, it may be of some *psychological* or *literary* interest. After all literary scholars often go to great lengths to try and understand 'what the great poet had in mind', and most often get it wrong. Often the poet had nothing in mind, and if she did, it was something very mundane.

Parody

Let's use an analogy. Suppose that in the American Mathematical Monthly analog of Egypt in 5000BC there was a papyrus that gives an elegant proof of the identity

$$123\cdot 321\;=\;39483\quad, \tag{1}$$

and then it comments that it is equivalent to

$$111\cdot 449\;=\;49839\quad, \tag{2}$$

but that the latter identity is no easier to prove than the former one. In a logical sense, that ancient savant would have been right, since all correct statements are logically equivalent. But some mathematical historian could have been tempted to 'explain' how Professor Ahmes may have reasoned. Since $111=(123-12)$ and $449=321+128$ (proofs left to the reader), we have

$$111\cdot 449\;=\;(123-12)(321+128)\;=\;123\cdot 321+123\cdot 128-12\cdot 321-12\cdot 128$$

$$=123\cdot 321+15744-3582-1536\quad.$$

[1] There is still room for **elegant** and insightful combintorial proofs of such identities, of course, see [MTWZ] for two such gorgeous proofs.

Using Eq. (1), the first product equals 39483, and it follows that indeed

$$111 \cdot 449 \;=\; 39483 + 15744 - 3852 - 1536 \;=\; 49839 \quad QED.$$

With all due respect, it is much easier to prove identities (1) and (2) separately, than to only prove (1) and then to deduce (2) from it.

Postscript (Added March 23, 2019)

It turns out that the two sequences, $A(n) := \int_0^1 (3x^2 - 2x^3)^n \, dx$ and $B(n) := \sum_{j=0}^{n} 2^j \binom{3n+1}{n-j}$ (that equals the other three binomial coefficients sums), **are** related. In fact

$$A(n) = \frac{(2n)!n!}{(3n+1)!} \, B(n) \quad .$$

This follows by 'solving' the two first-order linear inhomogeneous recurrences derived above for $A(n)$ and $B(n)$, in terms of **indefinite** sums ($A(n)$ is defined as a *definite* integral and $B(n)$ as a *definite* sum).

Solving the recurrences $9(n+1)(2n+1)A(n) \,-\, 2(3n+4)(3n+2)A(n+1) \;=\; 1$ and $-27\, B\,(n) + 4\, B\,(n+1) \;=\; -3\, \frac{(3n+1)!}{(2n+1)!(n+1)!}$ derived above yields that

$$B(n) = (\frac{27}{4})^n \left(1 - \sum_{k=1}^{n} (\frac{4}{27})^k \frac{(3k-2)!}{(2k-1)!k!} \right) \quad ,$$

with a similar formula for $A(n)$ that proves that $A(n)/B(n) = \frac{(2n)!n!}{(3n+1)!}$. Alternatively, performing the **dependent change of variable** $A(n) = \frac{(2n)!n!}{(3n+1)!} \, B(n)$ on the recurrence for $A(n)$ yields the recurrence for $B(n)$.

PostPostscript (Added March 24, 2019) In hindsight, the binomial coefficients identity

$$\sum_{j=0}^{2n} (-4)^j \binom{3n+1}{n+j+1} \;=\; \sum_{j=0}^{n} 2^j \binom{3n+1}{n-j} \quad , \qquad (RuehrSum)$$

is 'equivalent' to the integral identity

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} (3x^2 - 2x^3)^n \, dx \;=\; \int_0^1 2\,(3x^2 - 2x^3)^n \, dx \quad . \qquad (RuehrIntegral)$$

Multiplying both sides of $(RuehrSum)$ by $\frac{(2n)!n!}{(3n+1)!}$, simplifying, replacing $\frac{n!j!}{(n+j+1)!}$ in the left sum and $\frac{(2n)!j!}{(2n+j+1)!}$ in the right sum by $\int_0^1 x^n (1-x)^j$ and $\int_0^1 x^{2n}(1-x)^j$ respectively, and using the binomial theorem, leads to the following integral identity

$$\int_0^1 x^n (4x-3)^{2n} \, dx \;=\; \int_0^1 x^{2n} (3-2x)^n \, dx \quad ,$$

that is presuambly equivalent to ($RuehrIntegral$).

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