REVEREND CHARLES to the aid of MAJOR PERCY and FIELDS-MEDALIST ENRICO

Doron ZEILBERGER[1]

Voltaire said that Archimedes had more imagination than Homer. Unfortunately, as far as we know, Archimedes's only use of it, outside of mathematics, was to condemn goldsmiths and to kill people. More peaceful uses of mathematicians' imagination, to the outside of mathematics, in decreasing order of impact, were provided by Multi-Millionaire Richard Garfield, the Reverend Charles Dodgson, and Major Percy MacMahon, who respectively developed : 'Magic: The Gathering' (*the* game of our decade), Alice, and an earlier version of Instant Insanity.

This is not to say that their imagination did not also help mathematics proper. In this *quickie*, I observe how Dodgson's[D] rule for evaluating determinants: (For any $n \times n$ matrix A, let $A_r(k,l)$ be the $r \times r$ submatrix whose upper leftmost corner is the entry $a_{k,l}$,)

$$\det A \;=\; \frac{\det A_{n-1}(1,1)\det A_{n-1}(2,2) - \det A_{n-1}(1,2)\det A_{n-1}(2,1)}{\det A_{n-2}(2,2)} \quad ,$$

immediately implies MacMahon's[M] determinant evaluation:

$$\det \left[\binom{a+i}{b+j}_{1 \le i,j \le n} \right] = \frac{(a+n)!!(n-1)!!(a-b-1)!!(b)!!}{(a)!!(a-b+n-1)!!(b+n)!!} \quad ,$$

where, $n!! := 1!2!3! \cdots n!$, and, of course, $n! := 1 \cdot 2 \cdots n$.

Indeed, let the left and right sides be $L_n(a,b)$ and $R_n(a,b)$ respectively. Dodgson's rule immediately implies that the recurrence:

$$X_n(a,b) = \frac{X_{n-1}(a,b)X_{n-1}(a+1,b+1) - X_{n-1}(a+1,b)X_{n-1}(a,b+1)}{X_{n-2}(a+1,b+1)} \quad ,$$

holds with $X = L$. Since $L_n(a,b) = R_n(a,b)$ for $n = 0, 1$ (check!), and the recurrence also holds with $X = R$ (check![2]), it follows by induction that $L_n(a,b) = R_n(a,b)$ for *all* n. $\Box$.

The special case $a = 2n+1$, and $b = n$, reduces to a special case of a conjecture of Enrico Bombieri[3], David Hunt, and Alf van der Poorten ([BHP]). While this special case was already done in [BHP] using a different method, I believe that Dodgson's method should be extendable to prove their full conjecture.

1 Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `E-mail:zeilberg@math.temple.edu;` `WWW: http://www.math.temple.edu/~zeilberg; Ftp: ftp.math.temple.edu, directory /pub/zeilberg.`
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2 Divide both sides by the left, then use r!!/(r-1)!!=r! whenever possible, and then r!/(r-1)!=r whenever possible, reducing it to a completely routine polynomial identity.

3 Who applies his imagination not only to mathematics, but also to art: he is an accomplished painter.

References

[BHP] E. Bombieri, D.C. Hunt, and A.J. van der Poorten, *Determinants in the study of Thue's method and curves with prescribed singularities*, J. Experimental Mathematics, to appear.

[D] C.L. Dodgson, *Condensation of Determinants*, Proceedings of the Royal Society of London **15**(1866), 150-155.

[M] P.A. MacMahon, *"Combinatory Analysis"*, Cambridge University Press, 1918. [reprinted by Chelsea, 1984].