Using Noonan-Zeilberger Functional Equations to enumerate (in Polynomial Time!) Generalized Wilf classes

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Very Important

This article is accompanied by the Maple packages

- `http://www.math.rutgers.edu/~zeilberg/tokhniot/P123` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/F123` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/P1234` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/F1234` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/P12345` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/F12345` ,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/P123456` ,

to be described below. Lots of sample input and output files can be seen at:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/Gwilf.html` .

Introduction

Recall that the *reduction* of a finite list of k, say, distinct (real) numbers $[a_1, a_2, \ldots, a_k]$ is the unique permutation $\sigma = [\sigma_1, \ldots, \sigma_k]$, of $\{1, \ldots, k\}$ such that a_1 is the σ_1-th largest element in the list, a_2 is the σ_2-th largest element in the list, etc. In other words $[a_1, a_2, \ldots, a_k]$ and σ are "order-isomorphic". For example, the reduction of $[6, 3, 8, 2]$ is $[3, 2, 4, 1]$ and the reduction of $[\pi, \gamma, e, \phi]$ is $[4, 1, 3, 2]$ (where ϕ is the Golden Ratio).

Given a permutation $\pi = \pi_1 \ldots \pi_n$ and another permutation $\sigma = [\sigma_1, \ldots, \sigma_k]$ (called a *pattern*), we denote by $N_\sigma(\pi)$ the number of instances $1 \le i_1 < \ldots < i_k \le n$ such that the reduction of $\pi_{i_1} \ldots \pi_{i_k}$ is σ.

For example, if $\pi = 51324$ then

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$N_{[1,2,3]}(\pi) = 2$ (because $\pi_2\pi_3\pi_5 = 134$ and $\pi_2\pi_4\pi_5 = 124$ reduce to $[1, 2, 3]$).

$N_{[1,3,2]}(\pi) = 1$ (because $\pi_2\pi_3\pi_4 = 132$ reduces to $[1, 3, 2]$).

$N_{[2,1,3]}(\pi) = 1$ (because $\pi_3\pi_4\pi_5 = 324$ reduces to $[2, 1, 3]$).

$N_{[2,3,1]}(\pi) = 0$ (because none of the 10 length-three subsequences of π reduces to 231).

$N_{[3,1,2]}(\pi) = 5$ (because $\pi_1\pi_2\pi_3 = 513$ and $\pi_1\pi_2\pi_4 = 512$ and $\pi_1\pi_2\pi_5 = 514$ and $\pi_1\pi_3\pi_5 = 534$ and $\pi_1\pi_4\pi_5 = 524$ all reduce to $[3, 1, 2]$) .

$N_{[3,2,1]}(\pi) = 1$ (because $\pi_1\pi_3\pi_4 = 532$ reduces to $[3, 2, 1]$).

Of course the sum of $N_\sigma(\pi)$ over all k-permutations σ is $\binom{n}{k}$.

Fixing a pattern σ, the set of permutations π for which $N_\sigma(\pi) = 0$ (we say that π *avoids* σ) is called the *Wilf class* of σ, and more generally, given a set of patterns S, the set of permutations for which $N_\sigma(\pi) = 0$ for all $\sigma \in S$, is the Wilf class of that set. The first *systematic* study of *enumerating* Wilf classes was undertaken in the pioneering paper by Rodica Simion and Frank Schmidt [SiSc].

The general question is extremely difficult (see [Wiki] and [Bo3]) and "explicit" answers are only known for few short patterns (and sets of patterns), the increasing patterns $[1, 2, \ldots, k]$, and a few other *West-equivalent* to them, giving the same enumeration. For example, even for the pattern $[1, 3, 2, 4]$ (`http://oeis.org/A061552`) the best known algorithm takes exponential time in n, and it is very possible that that's the best that one can do.

But for those patterns σ for which we know how to enumerate their Wilf classes, most importantly the increasing patterns $[1, \ldots, k]$, it makes sense to ask the more general question:

Given a pattern σ and a positive integer r, find a "formula", or at least a polynomial-time algorithm (thus *answering* the question in the sense of Herb Wilf[Wil]) that inputs a positive integer n and outputs the number of permutations π of $\{1, \ldots, n\}$ for which $N_\sigma(\pi) = r$. We call such a class a *generalized* Wilf class.

Ideally, we would like to have, given a pattern σ, an explicit formula, in n and q, for the generating function (S_n denotes the set of permutations of $\{1, \ldots, n\}$)

$$A_\sigma(q, n) := \sum_{\pi \in S_n} q^{N_\sigma(\pi)} \quad ,$$

then, for any fixed r, the sequence of coefficients of q^r in $A_\sigma(q, n)$ would give the sequence enumerating permutations with *exactly* r occurrences of the pattern σ.

In fact, for patterns of length ≤ 2 there are nice answers. Trivially

$$A_{[1]}(q, n) := n!\, q^n \quad ,$$

and almost-trivially (or at least classically)

$$A_{[2,1]}(q,n) := (1)\,(1+q)\,\cdots\,(1+q+\ldots+q^{n-1}) \;=\; [n]! \quad ,$$

the famous "q-analog" of $n!$. But things start to get complicated for patterns of length 3.

Past Work

For a very lucid and extremely engaging introduction to the subject, as well as the state-of-the-art, we strongly recommend Miklós Bóna's *masterpiece* [Bo3].

In [NZ], John Noonan and the second-named author initiated a *functional equations*-based approach for enumerating generalized Wilf classes. In order to illustrate it, they reproved John Noonan's[N] combinatorially-proved result that the number of permutations of length n with exactly *one* occurrence of the pattern $[1,2,3]$ equals $\frac{3}{n}\binom{2n}{n-3}$. Recently a *proof from the book* of this result was given by Alexander Burstein[Bu] (see also [Z1]).

In [NZ] it was conjectured that the number of permutations of length n with *two* occurrences of the pattern $[1,2,3]$ equals $\frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$. This conjecture was proved by Markus Fulmek[F], using Dyck paths. A very interesting, purely human, approach was developed by David Callan[C] who derived expressions for enumerating permutations of length n with *three* occurrences and *four* occurrences.

In [NZ] it was also conjectured that the number of permutations of length n with *one* occurrence of the pattern $[1,3,2]$ equals $\frac{n-2}{2n}\binom{2n-2}{n-1}$. This conjecture was proved by Miklós Bóna[Bo1], who later proceeded to prove[Bo2] the interesting fact that the sequences enumerating permutations with exactly r occurrences of $[1,3,2]$ is P-recursive (i.e. satisfies a homogeneous linear recurrence with polynomial coefficients) for *every* r. In fact he proved the stronger result that the generating functions are always algebraic. This was vindicated by Toufik Mansour and Alek Vainshtein[MV] who gave an efficient algorithm to actually compute these generating functions, and they used it to find explicit expressions for $1 \le r \le 5$.

Another interesting but different "functional equation" approach, for patterns of length three, was developed by Firro and Mansour[FM].

This Project

But so far, practically nothing is known for patterns of length larger than three and $r > 0$. In this paper we will modify the approach of [NZ] in order to generate, in *polynomial time*, such sequences for increasing patterns of *any* length $[1,\ldots,k]$. That method can be extended to the patterns $[1,\ldots,k-2,k,k-1]$ and possibly other families, but here we will only discuss increasing patterns.

Let us emphasize that the "brute force" approach requires exponential time, since we actually have to construct the set of permutations of length n with the given specifications, and then take the cardinality.

Using the new algorithm to compute sufficiently many terms, we were able to conjecture explicit formulas, in n, for the number of permutations of length n with exactly r occurrences of the pattern $[1,2,3]$, for $5 \leq r \leq 7$, extending Fulmek's[F] conjectures for $r=3$ and $r=4$. We believe that the *enumeration schemes*, that our algorithms generate, should enable our computers to conjecture *holonomic* representations for the more general quantities (see below), that once guessed, should be amenable to automatic rigorous proving in the *holonomic paradigm*[Z2], using Christoph Koutschan's[K] far-reaching extensions and powerful implementations. But since these conjectures are *certainly* true, and their formal proof would (probably) not yield any new insight, we don't think that it is worth the trouble to actually carry out the gory details, wasting both humans' time (it would require quite a bit of daunting programming) and the computers' time (it would take a very long time, due to the complexity of the schemes).

Now let's recall the *Noonan-Zeilberger Functional Equation Approach.*

The Noonan-Zeilberger Functional Equation Approach

The *starting point* of the Noonan-Zeilberger[NZ] approach for enumerating generalized Wilf classes is to derive a *functional equation.* Let's review it with the simplest non-trivial case, that of the length-3 increasing pattern $[1,2,3]$.

In addition to the variable q, introduce n extra *catalytic variables* $x_1, \ldots, x_n$, and define the *weight* of a permutation $\pi = \pi_1 \ldots \pi_n$ of length n by

$$weight(\pi) := q^{N_{[1,2,3]}(\pi)} \prod_{i=1}^{n} x_i^{|\{1 \leq a < b \leq n; \pi_a = i < \pi_b\}|} \quad ,$$

(as usual, for any set A, $|A|$ denotes the number of elements of A). For example,

$$weight(12345) = q^{10} x_1^4 x_2^3 x_3^2 x_4 \quad ,$$

$$weight(54321) = 1 \quad ,$$

$$weight(21354) = q^4 x_2^3 x_1^3 x_3^2 = q^4 x_1^3 x_2^3 x_3^2 \quad .$$

Let's define the polynomial in the $n+1$ variables

$$P_n(q; x_1, \ldots, x_n) := \sum_{\pi \in S_n} weight(\pi) \quad .$$

Let $\pi = \pi_1 \ldots \pi_n$ be a typical permutation of length n. Suppose $\pi_1 = i$. Note that the number of occurrences of the pattern $[1,2,3]$ in π equals the number of occurrences of that pattern in the beheaded permutation $\pi_2 \ldots \pi_n$ **plus** the number of the patterns $[1,2]$ in the beheaded permutation $\pi_2 \ldots \pi_n$ where the "1" is $i+1$, or $i+2$, or $\ldots$ or n. Let π' be the reduction to $\{1, \ldots, n-1\}$ of that beheaded permutation. We see that

$$weight(\pi) = x_i^{n-i} weight(\pi') \,|_{x_i \to q x_{i+1} \,,\, x_{i+1} \to q x_{i+2}, \,\ldots\,,\, x_{n-1} \to q x_n} \quad .$$

The factor of x_i^{n-i} is because converting π' from a permutation of $\{1,\dots,n-1\}$ to a permutation of $\{1,\dots,i-1,i+1,\dots,n\}$, and sticking an i at the front introduces $n-i$ new $[1,2]$ patterns where the "1" is i. This gives the *Noonan-Zeilberger Functional Equation*

$$P_n(q;x_1,\dots,x_n)=\sum_{i=1}^{n} x_i^{n-i} P_{n-1}(q;x_1,\dots,x_{i-1},qx_{i+1},\dots,qx_n) \quad . \qquad (NZFE1)$$

Having found $P_n(q;x_1,\dots,x_n)$, we set the "catalytic" variables $x_1,\dots,x_n$ all to 1 and get

$$f_n(q):=A_{[1,2,3]}(q,n)=P_n(q;1,1,\dots,1) \quad .$$

Even though this is an "exponential-time" (and memory!) algorithm, it is much faster than the direct weighted counting of all the $n!$ permutations, and we were able to explicitly compute them through $n=20$.

This is implemented in procedure `fn(n,q)` in `P123`. Procedure `L20(q);` gives the pre-computed sequence of `fn(n,q)` for n between 1 and 20 .

Here are the first few terms:

$$f_1(q)=1 \quad , \quad f_2(q)=2 \quad , \quad f_3(q)=q+5 \quad , \quad f_4(q)=q^4+3\,q^2+6\,q+14 \quad ,$$

$$f_5(q)=q^{10}+4\,q^7+6\,q^5+9\,q^4+7\,q^3+24\,q^2+27\,q+42 \quad ,$$

$$f_6(q)=q^{20}+5\,q^{16}+8\,q^{13}+6\,q^{12}+6\,q^{11}+16\,q^{10}+12\,q^9+24\,q^8+32\,q^7+37\,q^6+54\,q^5+74\,q^4+70\,q^3+133\,q^2+110\,q+132 \quad ,$$

$$f_7(q)=q^{35}+6\,q^{30}+10\,q^{26}+10\,q^{25}+8\,q^{23}+13\,q^{22}+30\,q^{21}+10\,q^{20}+32\,q^{19}+18\,q^{18}+62\,q^{17}+74\,q^{16}+24\,q^{15}+100\,q^{14}$$
$$+130\,q^{13}+104\,q^{12}+162\,q^{11}+191\,q^{10}+232\,q^9+260\,q^8+320\,q^7+387\,q^6+395\,q^5+507\,q^4+461\,q^3+635\,q^2+429\,q+429 \quad ,$$

$$f_8(q)=q^{56}+7\,q^{50}+12\,q^{45}+15\,q^{44}+10\,q^{41}+16\,q^{40}+40\,q^{39}+18\,q^{38}+47\,q^{36}+38\,q^{35}+68\,q^{34}+60\,q^{33}$$
$$+58\,q^{32}+66\,q^{31}+154\,q^{30}+138\,q^{29}+115\,q^{28}+156\,q^{27}+252\,q^{26}+324\,q^{25}+228\,q^{24}+288\,q^{23}+537\,q^{22}$$
$$+466\,q^{21}+546\,q^{20}+656\,q^{19}+682\,q^{18}+1004\,q^{17}+1047\,q^{16}+886\,q^{15}+1494\,q^{14}+1456\,q^{13}+1580\,q^{12}$$
$$+1818\,q^{11}+2077\,q^{10}+2182\,q^9+2389\,q^8+2544\,q^7+2864\,q^6+2570\,q^5+3008\,q^4+2528\,q^3+2807\,q^2+1638\,q+1430 \quad .$$

For $f_n(q)$ for $9 \leq n \leq 20$ see:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oP123d` .

Using this data, the computer easily finds *rigorously-proved* explicit expressions for the first six moments (about the mean) of the random variable "number of occurrences of the pattern $[1, 2, 3]$", and from them verifies that, at least up to the sixth moment, this random variable is *asymptotically normal*, as humanly proved (for *all* patterns) by Miklós Bóna[Bo4]. See:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oP123a` .

The "Perturbation" Approach

The equations of *quantum field theory* are (usually) impossible to solve exactly, but physicists got around it by devising clever "approximate" methods using *perturbation* expansions, that only use the first few terms in a potentially "infinite" (and intractable) series, but that suffice for all practical purposes, using *Feynman diagrams*.

Of course, we are *enumerators*, and we want *exact* results, but suppose we only want to know the sequences enumerating permutations with exactly s occurrences of the pattern $[1, 2, 3]$ for $s \leq r$ for some relatively small r, rather than for $r = \binom{n}{3}$, provided by the full

$$f_n(q) = P_n(q\,;\,1\,[n \;\; times\,]) \quad .$$

In the original article [NZ], for $r = 0$, Noonan and Zeilberger simply plugged-in $q = 0$ and $x_1 = \ldots = x_n = 1$, getting a simple enumeration scheme, that proved, for the n-th time, the classical result that the number of permutations of length n that avoid the pattern 123 equals the Catalan number $(2n)!/(n!(n+1)!)$. For $r = 1$, they differentiated Eq. $(NZFE1)$ with respect to q, using the multivariable calculus *chain rule*, and then plugged-in $q = 0$ and $x_1 = \ldots = x_n = 1$. For $r = 2$ they did it again, but this turned out to be, for larger r, a *Rube Goldberg* nightmare, *even for a computer*.

Here is a much easier way!

Recall that you are really only interested in $f_n(q) = P_n(q\,;\,1\,[n \;\; times]\,)$. Plugging it into (NZFE1) gives

$$P_n(q\,;\,1\,[n \;\; times\,]) \; = \; \sum_{i=1}^{n} P_{n-1}(q\,;\,1\,[\,i-1 \;\; times\,]\,,\,q\,[\,n-i \;\; times\,]) \quad .$$

This forces us to put-up with expressions of the form

$$P_{a_0+a_1}(q;\,1\,[\,a_0 \;\; times]\,,\,q\,[\,a_1 \;\; times]) \quad .$$

Plugging this into $(NZFE1)$ yields

$$P_{a_0+a_1}(q;\,1\,[\,a_0 \;\; times]\,,\,q\,[\,a_1 \;\; times]) = \sum_{i=1}^{a_0} P_{a_0+a_1-1}(q;\,1\,[\,i{-}1 \;\; times]\,,\,q\,[\,a_0{-}i \;\; times]\,,\,q^2[a_1 \;\; times])$$

$$+\sum_{i=1}^{a_1} q^{a_1-i} P_{a_0+a_1-1}(q;\, 1\,[\,a_0\ \ times]\,,\, q\,[\,i-1\ \ times]\,,\, q^2\,[\,a_1-i\ \ times]) \quad .$$

This forces us, in turn, to consider expressions of the form

$$P_{a_0+a_1+a_2}(q;\, 1\,[\,a_0\ \ times]\,,\, q\,[\,a_1\ \ times]\,,\, q^2\,[\,a_2\ \ times]),$$

that would force us to further consider expressions of the form

$$P_{a_0+a_1+a_2+a_3}(q;\, 1\,[\,a_0\ \ times\,]\,,\, q\,[\,a_1\ \ times]\,,\ q^2\,[\,a_2\ \ times]\,,\, q^3\,[\,a_3\ \ times]) \quad ,$$

etc. etc., leading to an **exponential explosion** in both time and memory.

But, if we are only interested in the first r coefficients of $f_n(q)$, then we can take advantage of the **crucial** lemma, that follows immediately from the definition of the weight.

Crucial Lemma: For $s > r+1$, the coefficients of $q^0, q^1, \dots, q^r$ of

$$P_{a_0+a_1+\ldots+a_s}(q;\, 1\,[\,a_0\ \ times\,]\,,\, \ldots q^{s-1}\,,\, [\,a_{s-1}\ \ times\,]\,,\, q^s\,[\,a_s\ \ times\,])$$

$$-\, P_{a_0+a_1+\ldots+a_s}(q;\, 1\,[\,a_0\ \ times]\,,\, \ldots\,,\, q^r\,[a_r\ \ times\,]\,,\, q^{r+1}\,[\,a_{r+1}+a_{r+2}+\ldots+a_s\ \ times\,])$$

all vanish.

Let $n := a_1 + \ldots + a_r + a_{r+1}$. For any expression R and positive integer k, let $R\$k$, denote $R \ldots R[k\ \ times]$ for example $q^2\$3$ means q^2, q^2, q^2. Also, for any polynomial $p(q)$ in q, let $p^{(r)}(q)$ denote the polynomial of degree r obtained by ignoring all powers of q larger than r, and let $CHOP_r[\,p(q)\,] := p^{(r)}(q)$.

Eq. $(NZFE1)$ becomes

$$P_n^{(r)}(q;\, 1\$a_0\,,\, q\$a_1\,,\, \ldots\,,\, q^r\$a_r, q^{r+1}\$a_{r+1})$$

$$= CHOP_r\,[\sum_{i=1}^{a_0} P_{n-1}^{(r)}(q;\, 1\$(i-1)\,,\, q\$(a_0-i)\,,\, q^2\$a_1\,,\, \ldots,\ q^r\$a_{r-1}\,,\, q^{r+1}\$(a_r+a_{r+1}))$$

$$+\sum_{i=1}^{a_1} q^{a_1-i+a_2+\ldots+a_{r+1}} P_{n-1}^{(r)}(q;\, 1\$a_0\,,\, q\$(i-1)\,,\, q^2\$(a_1-i)\,,\, \ldots\,,\, q^r\$a_{r-1}\,,\, q^{r+1}\$(a_r+a_{r+1}))$$

$$+\sum_{i=1}^{a_2} q^{2(a_2-i+a_3+\ldots+a_{r+1})} P_{n-1}^{(r)}(q;\, 1\$a_0\,,\, q\$a_1\,,\, q^2\$(i-1), q^3\$(a_2-i)\,,\, \ldots\,,\, q^r\$a_{r-1}\,,\, q^{r+1}\$(a_r+a_{r+1}))$$

$$+ \ldots\ldots\ldots$$

$$+\sum_{i=1}^{a_{r+1}} q^{(r+1)(a_{r+1}-i)} P_{n-1}^{(r)}(q;\, 1\$a_0\,,\, q\$a_1\,,\, q^2\$a_2\, \ldots\,, q^r\$a_r\,,\ q^{r+1}\$(a_{r+1}-1))\,] \quad .$$

Now note that, because of the $CHOP_r$ operator in front, many terms automatically disappear, because of the powers of q in front. The *bottom line* is that the computer can automatically generate a scheme for computing the degree-r polynomials in q,

$$F_r(a_0,\dots,a_{r+1})(q) \;:=\; P^{(r)}_{a_0+\ldots+a_{r+1}}(q;\,1\,[\,a_0\;\;times\,]\,,\,q\,[\,a_1\;\;times]\,,\,\dots\,,\,q^{r+1}\,[\,a_{r+1}\;\;times\,]) \quad ,$$

with $a_0+\ldots+a_{r+1}=n$ and $a_0,\dots,a_{r+1}\geq 0$. The number of such quantities is the coefficient of z^n in $1/(1-z)^{r+2}$ that equals $(-1)^{r+2}\binom{-(r+2)}{n}=\binom{r+n+1}{r+1}$ terms. So each iteration involves $O(n^{r+1})$ evaluations and hence $O(n^{r+2})$ additions and doing it n times yields an $O(n^{r+3})$ algorithm for finding our object of desire, the degree r polynomial in q:

$$f^{(r)}_n(q) \;=\; F_r(n\,,\,0\,[\,r+1\;\;times\,])(q) \quad .$$

Having found the scheme, the very same computer (or a different one), may use it to generate as many terms as desired.

The Maple package P123

The Maple package P123 downloadable from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/P123` ,

implements the functional equation $(NZFE1)$ and easily generated the first 25 terms of the enumerating sequences for $0\leq r\leq 7$. With this data, it empirically verified the already-known results for the number of permutations with exactly r occurrences of the pattern $[1,2,3]$ for $0\leq r\leq 4$ (due to Noonan[N] $(r=1)$, Fulmek[F] $(r=2)$, and Callan[C] $(r=3,4)$, and made conjectures for $5\leq r\leq 7$ as follows. Let $a_r(n)$ be the number of permutations of length n with exactly r occurrences of the pattern $[1,2,3]$.

$$a_0(n)=2\,\frac{(2\,n-1)!}{(n-1)!\;(n+1)!} \quad .$$

$$a_1(n)=6\,\frac{(2\,n-1)!}{(n-3)!\;(n+3)!} \quad .$$

$$a_2(n)=\frac{(2\,n-2)!}{(n-4)!\;(n+5)!}\cdot(59\,n^2+117\,n+100) \quad .$$

$$a_3(n)=\frac{(2\,n-3)!}{(n-5)!\;(n+7)!}\cdot 4\,n\left(113\,n^3+506\,n^2+937\,n+1804\right) \quad .$$

$$a_4(n) = \frac{(2n-4)!}{(n-4)!(n+9)!} \cdot$$

$$\left(3561\,n^8 + 3126\,n^7 - 46806\,n^6 + 12384\,n^5 - 659091\,n^4 + 2630634\,n^3 + 5520576\,n^2 + 26283456\,n - 39191040\right) \quad .$$

$$a_5(n) = \frac{(2n-5)!}{(n-5)!(n+11)!} \cdot$$

$$(\,26246\,n^{10} + 136646\,n^9 - 115872\,n^8 + 22524\,n^7 - 9648450\,n^6 + 71304534\,n^5$$

$$+381205612\,n^4 + 1607633896\,n^3 + 2800103664\,n^2 + 3611692800\,n - 32891443200) \quad .$$

$$a_6(n) = \frac{(2n-6)!}{(n-6)!(n+13)!} \cdot$$

$$(\,193311\,n^{12} + 2349954\,n^{11} + 13035003\,n^{10} + 95151030\,n^9 + 406430793\,n^8 + 2889552582\,n^7$$

$$+14335663329\,n^6 + 60005854890\,n^5 + 313010684796\,n^4 + 1025692693464\,n^3$$

$$+1283595375168\,n^2 - 6909513045120\,n - 28177269120000\,) \quad .$$

$$a_7(n) = \frac{(2n-7)!}{(n-5)!(n+15)!} \cdot$$

$$(\,1386032\,n^{16} + 13111080\,n^{15} + 22526480\,n^{14} + 355187760\,n^{13} - 1654450096\,n^{12} + 10534951680\,n^{11}$$

$$+15797223760\,n^{10} - 305671694640\,n^9 + 3750695521216\,n^8 - 26631101348520\,n^7$$

$$-86395090065440\,n^6 - 636425872408320\,n^5 + 3647384624274048\,n^4$$

$$+11386434230674560\,n^3 + 103032675524966400\,n^2 - 157858417817856000\,n - 763734137886720000\,) \quad .$$

Enumerating Permutations with r occurrences of the pattern [1,2,3,4] for small r via a Noonan-Zeilberger Functional Equation

In addition to the variable q, we now introduce $2n$ extra *catalytic variables* $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$, and define the *weight* of a permutation $\pi = \pi_1 \ldots \pi_n$ of length n by

$$weight(\pi) := q^{N_{[1,2,3,4]}(\pi)} \prod_{i=1}^{n} x_i^{|\{1 \le a < b < c \le n \,;\, \pi_a = i < \pi_b < \pi_c\}|} \cdot y_i^{|\{1 \le a < b \le n \,;\, \pi_a = i < \pi_b\}|} \quad .$$

For example,

$$weight([1,2,3,4,5,6]) = q^{15} x_1^{10} x_2^6 x_3^3 x_4 y_1^5 y_2^4 y_3^3 y_4^2 y_5 \quad ,$$

$$weight([6,5,4,3,2,1]) = 1 \quad ,$$

$$weight([3,4,5,6,1]) = q x_3^3 x_4 y_3^3 y_4^2 y_5 \quad .$$

Let us define the polynomial in the $2n+1$ variables

$$P_n(q; x_1, \dots, x_n; y_1, \dots, y_n) := \sum_{\pi \in S_n} weight(\pi) \quad .$$

Let $\pi = \pi_1 \dots \pi_n$ be a typical permutation of length n. Suppose $\pi_1 = i$. Note that the number of occurrences of the pattern $[1,2,3,4]$ in π equals the number of occurrences of that pattern in the beheaded permutation $\pi_2 \dots \pi_n$ **plus** the number of the patterns $[1,2,3]$ in the beheaded permutation $\pi_2 \dots \pi_n$ where the "1" is $i+1$, or $i+2$, or ... or n. Let π' be the reduction to $\{1, \dots, n-1\}$ of that beheaded permutation. Also note that the number of occurrences of the pattern $[1,2,3]$ where the "1" is an i gets increased by the number of occurrences of the pattern $[1,2]$ in the beheaded permutation, where the "1" is a j with $j > i$. We see that

$$weight(\pi) = y_i^{n-i} weight(\pi') \mid_{x_i \to q x_{i+1} \ , \ x_{i+1} \to q x_{i+2}, \ \dots \ , \ x_{n-1} \to q x_n \quad ; \quad y_i \to x_i y_{i+1} \ , \ y_{i+1} \to x_i y_{i+2}, \ \dots \ , \ y_{n-1} \to x_i y_n} \quad .$$

The factor of y_i^{n-i} is because converting π' from a permutation of $\{1, \dots, n-1\}$ to a permutation of $\{1, \dots, i-1, i+1, \dots, n\}$, and sticking an i at the front introduces $n-i$ new $[1,2]$ patterns where the "1" is i. This gives the *Noonan-Zeilberger Functional Equation* for the pattern $[1,2,3,4]$:

$$P_n(q; x_1, \dots, x_n \, ; \, y_1, \dots, y_n) = \sum_{i=1}^{n} y_i^{n-i} P_{n-1}(q; x_1, \dots, x_{i-1}, q x_{i+1}, \dots, q x_n \, ; \, y_1, \dots, y_{i-1}, x_i y_{i+1}, \dots, x_i y_n) \quad . \tag{NZFE2}$$

Having found $P_n(q; x_1, \dots, x_n \, ; \, y_1, \dots, y_n)$, we set the "catalytic" variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$ all to 1 and get

$$g_n(q) := A_{[1,2,3,4]}(q, n) = P_n(q; 1, 1, \dots, 1 \, ; \, 1, 1, \dots, 1) \quad .$$

Even though this is an "exponential-time" (and memory!) algorithm, it is still faster than the direct weighted counting of all the $n!$ permutations, and we were able to explicitly compute them through $n = 10$.

The first few polynomials are

$$g_1(q) = 1 \quad , \quad g_2(q) = 2 \quad , \quad g_3(q) = 6 \quad , \quad g_4(q) = q + 23 \quad ,$$

$$g_5(q) = q^5 + 4\,q^2 + 12\,q + 103 \quad ,$$

$$g_6(q) = q^{15} + 5\,q^9 + 8\,q^6 + 12\,q^5 + 6\,q^4 + 10\,q^3 + 63\,q^2 + 102\,q + 513 \quad ,$$

$$g_7(q) = q^{35} + 6\,q^{25} + 10\,q^{19} + 18\,q^{16} + 12\,q^{15} + 13\,q^{13} + 24\,q^{11} + 32\,q^{10} + 72\,q^9 + 10\,q^8 + 46\,q^7$$

$$+142\,q^6 + 116\,q^5 + 146\,q^4 + 196\,q^3 + 665\,q^2 + 770\,q + 2761 \quad ,$$

$$g_8(q) = q^{70}+7\,q^{55}+12\,q^{45}+15\,q^{41}+10\,q^{39}+8\,q^{36}+28\,q^{35}+40\,q^{32}+41\,q^{29}+10\,q^{28}+24\,q^{27}+44\,q^{26}+84\,q^{25}$$
$$+24\,q^{24}+89\,q^{23}+12\,q^{21}+142\,q^{20}+136\,q^{19}+96\,q^{18}+115\,q^{17}+333\,q^{16}+156\,q^{15}+112\,q^{14}+312\,q^{13}$$
$$+199\,q^{12}+600\,q^{11}+573\,q^{10}+804\,q^{9}+503\,q^{8}+885\,q^{7}+1782\,q^{6}+1204\,q^{5}+2148\,q^{4}+2477\,q^{3}+5982\,q^{2}+5545\,q+15767 \quad .$$

For $g_9(q), g_{10}(q)$ see: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oP1234d`.

The obvious analog of the Crucial Lemma still holds, and one can get *polynomial time* (in n) algorithms, to compute the number of permutations of length n with exactly r occurrences of the pattern $[1,2,3,4]$. Alas, because we have twice as many catalytic variables, the $O(n^{r+3})$ becomes $O(n^{2r+5})$. Nevertheless, we were able to compute the first 70 terms for the case $r=1$. Here are the first 23 terms:

$$0, 0, 0, 1, 12, 102, 770, 5545, 39220, 276144, 1948212, 13817680, 98679990,$$
$$710108396, 5150076076, 37641647410, 277202062666, 2056218941678, 15358296210724,$$
$$115469557503753, 873561194459596, 6647760790457218, 50871527629923754 \quad .$$

The rest can be viewed in: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oF1234a`.

Manuel Kauers has programmed our algorithm in the numeric language C, and used clever programming techniques to extend the table to 200 terms! We thank him dearly for allowing us to post them. These can be viewed here

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oF1234bManuelKauers`.

The Maple package P1234

Everything is implemented in the Maple package `P1234`. See the webpage of this article for sample input and output files. The package `F1234` is a more efficient implementation for a small number of occurrences r. Typing `Seq1();` in the Maple package `F1234` would give the first 200 terms, of the enumerating sequence for permutations with exactly one occurrence of the pattern $[1,2,3,4]$. As mentioned above, these were computed by Manuel Kauers.

Beyond

Of course, the same reasoning applies to any increasing pattern $[1,\ldots,k]$ but we have, in addition to q, $(k-2)n$ additional catalytic variables. For each specific r this implies a scheme that enables one to compute in "polynomial" time (in n, but of course not in k or r) the desired numbers. For the patterns $[1,2,3,4,5]$ and $[1,2,3,4,5,6]$ (i.e. $k=5$ and $k=6$) this is implemented in Maple packages `P12345` (and its more efficient [for small r] version `F12345`) and `P123456` respectively.

Other Patterns

Even the case of *pattern-avoidance*, i.e. $r = 0$, is already extremely difficult in general. As we mentioned above for the pattern $[1, 3, 2, 4]$, there is no known polynomial time algorithm for enumerating permutations that avoid it. In other words the best known algorithm that inputs a positive integer n and outputs the number of permutations of length n that avoid the pattern $[1, 3, 2, 4]$ takes exponentially many steps in n.

The Status of the Noonan-Zeilberger Conjecture

The Noonan-Zeilberger conjecture, made in [NZ], asserts that for any pattern, and for any positive integer r, the enumerating sequence of n-permutations with exactly r occurrences of that pattern is always P-recursive, i.e. satisfies a linear recurrence equation with polynomial coefficients in n. While still open, it seems much less likely to be true today than when it was first made almost twenty years ago. Manuel Kauers informed us, using computations modulo the prime 45007, that even 400 terms of the enumerating sequence for permutations with exactly one occurrence of $[1, 2, 3, 4]$ would not suffice to guess a recurrence. In other words, if a recurrence does exist, it would be extremely complicated. On the other hand, for $[1, 2, 3, 4]$-avoiding permutations, i.e. the case $r = 0$, there is a simple second-order recurrence.

Going back to the problem of plain enumeration, exact formulas (for the avoiding, $r = 0$, case) are known for a few infinite families (see [Bo3] and [Wiki]), and the present approach would hopefully be able to find polynomial-time schemes for $r > 0$, at least for some of them. We hope to investigate this in a future paper.

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