GAUSS'S ${}_2F_1(1)$ CANNOT BE GENERALIZED TO ${}_2F_1(x)$

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Abstract: Using ideas of Jet Wimp and Richard McIntosh, it is proved that Gauss's explicit evaluation of ${}_2F_1(a,b;c;1)$ cannot be generalized to ${}_2F_1(a,b;c;x)$, for arbitrary a,b,c, and x. A short proof of Wimp's theorem that asserts that ${}_3F_2(a,b,c;d,e;1)$ cannot be expressed in closed form is also given.

0 Introduction

As we all know, a *geometric series* is a series

$$\sum_{k=0}^{\infty} A_k \tag{1}$$

such that the ratio A_{k+1}/A_k of consecutive terms is identically equal to a *constant*, say r, and if $r < 1$, such a series is *explicitly summable*, the sum being $A_0/(1-r)$.

A *hypergeometric series* (e.g. [R], [B]) is a series (1) where A_{k+1}/A_k is a *rational function* of k. Writing this rational function in factored form

$$\frac{x(k+a_1)(k+a_2)...(k+a_p)}{(k+1)(k+b_1)(k+b_2)...(k+b_q)} \ , \tag{2}$$

the resulting series (1) (with $A_0 = 1$) is denoted by

$${}_pF_q\left(\begin{matrix} a_1 \ , \ \dots \ , \ a_p \\ b_1 \ , \ \dots \ , \ b_q \end{matrix} ; x\right) \ .$$

(The extra factor $(k+1)$ in the denominator of (2) is there for historical reasons, of course it is possible to get rid of it by making one of the numerator parameters a_i equal to 1.) In terms of the *raising factorial* $(a)_k := a(a+1)...(a+k-1)$, the general hypergeometric series can be written, and is usually defined by

$${}_pF_q\left(\begin{matrix} a_1 \ , \ \dots \ , \ a_p \\ b_1 \ , \ \dots \ , \ b_q \end{matrix} ; x\right) \ = \ \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{k!(b_1)_k...(b_q)_k} x^k \ .$$

There are many cases in which a hypergeometric series can be summed "explicitly", i.e. in terms of powers and products of Gamma functions. The simplest case is the binomial theorem:

$${}_1F_0\left(\begin{matrix} a \\ \_ \end{matrix} ; x\right) \ = \ (1-x)^{-a} \ . \tag{3}$$

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Moving up to ${}_2F_1$, we have Gauss's celebrated formula(e.g. [R], theorem 18, p.49):

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad . \tag{4} $$

Despite many attempts, no such formula for $F(a, b; c; x)$, with general x and general a, b, c was ever found, and the first purpose of the present paper is to prove that such a formula is impossible. Of course, for various specializations of x and a, b, c there do exist closed form formulas([B],[G-S]), the most well known being that of Kummer, in which $x = -1$ and $c = a - b + 1$([R], Theorem 26, p. 68):

$$ {}_2F_1\left(\begin{matrix} a, b \\ 1+a-b \end{matrix}; -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a/2-b)\Gamma(1+a)} \quad . \tag{5} $$

Another natural generalization of Gauss's formula would be an explicit expression for the general

$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; 1\right) \quad . \tag{6} $$

The celebrated formula of Pfaff-Saalschutz sums (6) ([R], p. 87) when one of the numerator parameters is a negative integer (so the series is *terminating*) and in addition it is *balanced*: $a_1 + a_2 + a_3 + 1 = b_1 + b_2$. This gives a four-parameter formula. The formula of Dixon([R], p. 92) sums (6) when it is *well-poised*: $a_1 + 1 = b_1 + a_2 = b_2 + a_3$, giving a 3-parameter formula. Other 3-parameter formulas are associated with the names of Watson and Whipple (see [B]). In addition there are many "strange" 2- and 1-parameter formulas conjectured by Gosper and proved by Gessel and Stanton[G-S]. The impossibility of a closed form evaluation of a general ${}_3F_2(a, b, c; d, e; 1)$ is a remarkable result due to Wimp[W]. Wimp used recurrences and an ingenious asymptotic argument. The second purpose of this paper is to give a short proof of Wimp's theorem. My proof uses Wimp's beautiful ideas, but the details are much shorter. I was also very much influenced by McIntosh's brilliant thesis[M] and his approach to proving minimality of recurrences.

1 ${}_2F_1(a, b; c; x)$ Is Not Nice

Theorem: There is no formula of the form

$$ {}_2F_1(a, b; c; x) = K\lambda^{A_0 a + B_0 b + C_0 c} \frac{\prod_{j=1}^{p} \Gamma(A_j a + B_j b + C_j c + D_j)}{\prod_{j=1}^{q} \Gamma(A'_j a + B'_j b + C'_j c + D'_j)} \ , \tag{7} $$

with A_0 , A_1 , ..., A_p , B_0 , B_1 , ..., B_p , C_0 , C_1 , ..., C_p, A'_1 , ..., A'_q , B'_0 , B'_1 , ..., B'_q , C'_1 , ..., C'_q all rational numbers, and K and λ are allowed to depend on x.

Proof: Suppose there is such a formula, then there would be a formula for

$$ a(n) := {}_2F_1(-n, n+1; 1; -1) = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k} \tag{8} $$

of the form

$$a(n) = K\lambda^{A_0 n}\frac{\prod_{j=1}^{p}\ \Gamma(A_j n + D_j)}{\prod_{j=1}^{q}\Gamma(A'_j n + D'_j)}\ , \tag{9}$$

with A_0, A_j, and A'_j rational numbers. Let K be the least common denominator of A_0, A_j, A'_j.Then obviously $a(n+K)/a(n)$ would be a rational function of n, i.e., there would be polynomials $P(n)$ and $Q(n)$ such that

$$P(n)a(n+K) - Q(n)a(n) \equiv 0, \quad n = 0, 1, 2...\ . \tag{10}$$

Note that $P(n)$ and $Q(n)$ can be chosen to have *integer* coefficients, since $a(n)$ are integers, and writing $P(n)$ and $Q(n)$ in generic form, plugging in the first few values of n, and solving the resulting system of linear equations, gives rational, and hence integer, solutions.

A routine application of the method of [K], pp. 66-67, shows that $a(n+1)/a(n)$ tends to $(1+\sqrt{2})^2$, as $n \to \infty$ (see also [P], p. 202). This implies that $a(n+K)/a(n)$ tends to $(1+\sqrt{2})^{2K}$ as $n \to \infty$. On the other hand if (10) were true, it would follow that if $a(n+K)/a(n)$ tends to any non-zero limit, that limit must be a rational number. Now $(1+\sqrt{2})^{2K}$ always has the form $a + b\sqrt{2}$, with a and b integers, and $b \neq 0$ (use the binomial theorem), and this must be irrational, thanks to Hippasus of Metapontum (see also [W']).

2 A Short Proof of Wimp's Theorem

I will now give a short proof of Wimp's theorem[W] that ${}_3F_2(a, b, c; d, e; 1)$ cannot be expressed in closed form, in the form analogous to (7). If it were then

$$b(n) := {}_3F_2(-n\,,\,-n\,,\,n+1\,;\,1\,,\,1\,;\,1) = \sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}^2 \tag{11}$$

would have a formula of the form (9), which would entail, for some integer K, and polynomials $P(n), Q(n)$ with integer coefficients that

$$P(n)b(n+K) - Q(n)b(n) \equiv 0, \quad n = 0, 1, 2...\ . \tag{12}$$

A routine application of the method of [K], pp. 66-67, shows that $b(n+1)/b(n)$ tends to $((1+\sqrt{5})/2)^5$, as $n \to \infty$ (see also [P], p. 200). The rest of the proof goes ditto, with $\sqrt{2}$ replaced by $\sqrt{5}$, and $2K$ replaced by $5K$.

Epilogue

The referee has found a much quicker proof of the main result of this paper. He pointed out that the specialization

$${}_2F_1(a, a+1/2; 1/2; x) = (1/2)(1+\sqrt{x})^{-2a} + (1/2)(1+\sqrt{x})^{2a}$$

also immediately implies that Gauss's evaluation of ${}_2F_1(1)$ cannot be extended to ${}_2F_1(x)$.

In spite of this beautiful proof of the referee, the present paper still serves a purpose. I believe that, although the referee calls his or her proof "trivial", it is very sleek, and I am glad that I can use the present paper to present it. Moreover, trivial or not, the result stated in the title is interesting, and ought to be pointed out. Finally, the *method* described in this paper, that was inspired by McIntosh and Wimp, can be used for proving non-closed-formness of many other hypergeometric and combinatorial sums.

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