The Jiang Zeng-M.Ishikawa-H.Tagawa q-Catalan Hankel Determinant Identity is Purely Routine

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Would you imagine a mathematical article spending 17 pages on several proofs of the identity $23 \cdot 21 = 483$? A first proof could be by explicitly drawing a rectangle of 23 by 21 dots, and asking the reader to count the number of dots. A more advanced proof could be

$$23 \cdot 21 = (20+3)\dot{(}20+1) = 20 \cdot 20 + 20 \cdot 1 + 3 \cdot 20 + 3 \cdot 1 = 400 + 20 + 60 + 3 = 400 + 80 + 3 = 483 \quad ,$$

and a really clever and elegant proof, using the advanced algebraic identity $(a-b)(a+b) = a^2 - b^2$ is as follows:

$$23 \cdot 21 = (22+1)\dot{(}22-1) = 22^2 - 1^2 = (2 \cdot 11)^2 - 1 = 4 \cdot 11^2 - 1 = 4 \cdot 121 - 1 = 484 - 1 = 483 \quad .$$

Of course not! *numerical* identities, and even *algebraic* identities (e.g. $(a+b)^2 = a^2 + 2ab + b^2$) and even *trig* identities (e.g. $\sin^2 x + \cos^2 x = 1$) are *nowadays* considered **routine**, since there exist **algorithms** for proving them (learned in third grade in the US and first grade in China).

Yet something analogous appeared in the recent article *"A q-analogue of Catalan Hankel determinants"* by M. Ishikawa, H. Tagawa, and Jiang Zeng, that was published in: *RIMS Kkyroku Bessatsu, B11 (2009), 19–42* and also posted in `http://arxiv.org/abs/1009.2004` . The main "theorem" (1.1) (or rather the more general corollary 1.2) follows *immediately* and **routinely** from Dodgson's condensation identity. Indeed calling the left side and right side of Eq. (1.14) of that paper $L(n,t)$ and $R(n,t)$ respectively, it follows, thanks to Rev. Charles, that $L(n,t) = (L(n-1,t)L(n-1,t+2) - L(n-1,t+1)^2)/L(n-2,t+2)$, and it is purely routine to check that the same identity holds with $L(n,t)$ replaced by $R(n,t)$, since this boils down to a certain routine polynomial identity in the variables a, b, q^n. Once this is done the "theorem" follows by induction since $L(0,t) = R(0,t)$ and $L(1,t) = R(1,t)$ (check!).

I recommend that the authors of this paper, and other people too, who wax insightful combinatorics on such routinely provable identities, read my article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/opa.html` ,

as well as the excellent paper by Tewodros Amdeberhan and myself:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/greg.html` . □

Added Dec. 23, 2011: Jiang Zeng just drew my attention to the fact that the above comment is actually mentioned in their paper! So they give several proofs to an identity that they *actually*

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`http://www.math.rutgers.edu/~zeilberg/` . Dec. 22, 2011. Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger `http://www.math.rutgers.edu/~zeilberg/pj.html` . Supported in part by the NSF.

knew was utterly trivial. They should have mentioned it in the abstract, and not bury it in a comment on p.12 .