The Jiang Zeng-M.Ishikawa-H.Tagawa q-Catalan Hankel Determinant Identity is Purely Routine

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Would you imagine a mathematical article spending 17 pages on several proofs of the identity $23 \cdot 21 = 483$? A first proof could be by explicitly drawing a rectangle of 23 by 21 dots, and asking the reader to count the number of dots. A more advanced proof could be

$$23 \cdot 21 = (20 + 3)(20 + 1) = 20 \cdot 20 + 20 \cdot 1 + 3 \cdot 20 + 3 \cdot 1 = 400 + 20 + 60 + 3 = 400 + 80 + 3 = 483,$$

and a really clever and elegant proof, using the advanced algebraic identity $(a - b)(a + b) = a^2 - b^2$ is as follows:

$$23 \cdot 21 = (22 + 1)(22 - 1) = 22^2 - 1^2 = (2 \cdot 11)^2 - 1 = 4 \cdot 11^2 - 1 = 4 \cdot 121 - 1 = 484 - 1 = 483.$$

Of course not! numerical identities, and even algebraic identities (e.g. $(a + b)^2 = a^2 + 2ab + b^2$) and even trig identities (e.g. $\sin^2 x + \cos^2 x = 1$) are nowadays considered routine, since there exist algorithms for proving them (learned in third grade in the US and first grade in China).

Yet something analogous appeared in the recent article “A q-analogue of Catalan Hankel determinants” by M. Ishikawa, H. Tagawa, and Jiang Zeng, that was published in: RIMS Kky-roku Bessatsu, B11 (2009), 19–42 and also posted in http://arxiv.org/abs/1009.2004. The main “theorem” (1.1) (or rather the more general corollary 1.2) follows immediately and routinely from Dodgson’s condensation identity. Indeed calling the left side and right side of Eq. (1.14) of that paper $L(n, t)$ and $R(n, t)$ respectively, it follows, thanks to Rev. Charles, that $L(n, t) = (L(n - 1, t)L(n - 1, t + 2) - L(n - 1, t + 1)^2)/L(n - 2, t + 2)$, and it is purely routine to check that the same identity holds with $L(n, t)$ replaced by $R(n, t)$, since this boils down to a certain routine polynomial identity in the variables $a, b, q^n$. Once this is done the “theorem” follows by induction since $L(0, t) = R(0, t)$ and $L(1, t) = R(1, t)$ (check!).

I recommend that the authors of this paper, and other people too, who wax insightful combinatorics on such routinely provable identities, read my article:

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/opa.html,

as well as the excellent paper by Tewodros Amdeberhan and myself:


Added Dec. 23, 2011: Jiang Zeng just drew my attention to the fact that the above comment is actually mentioned in their paper! So they give several proofs to an identity that they actually

knew was utterly trivial. They should have mentioned it in the abstract, and not bury it in a comment on p.12.