Two Quick Proofs of a Catalan Lemma Needed by Lisa Sauermann and Yuval Wigderson

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In memory of Robin Chapman (1963-2020), a problem-solving maestro and a great Catalanist

In a recent beautiful article [SW], the authors needed the following identity (Claim 3.8 there)

$$\sum_{i=0}^{s}(-1)^i\, C_i \binom{i+1}{s-i} \;=\; 0 \quad , \tag{1}$$

where C_i are the Catalan numbers. Let's give two shorter proofs of (1).

Proof 1 (by SBE): Go to Maple, and type [Note that one does **not** need Zeilberger, Gosper suffices].

```
sum((-1)**i*binomial(2*i,i)/(i+1)*binomial(i+1,s-i),i=0..s);
```
□

Proof 2 (by DZ) The Catalan number C_i famously (inter alia), counts the number of (complete) binary trees with $i+1$ leaves. $\binom{i+1}{s-i}$ counts the number of words with $s-i$ twos and $2i+1-s$ ones, hence the total number of words in the alphabet $\{1,2\}$ with $i+1$ letters whose sum is $s+1$.

Hence $\sum_{i=0}^{s} C_i \binom{i+1}{s-i}$ counts *all* binary trees with $\leq s+1$ leaves where the leaves have labels 1 or 2 that add-up to $s+1$. The lemma is equivalent to the fact that the number of such creatues with an **even** number of leaves equals the number of those with an **odd** number of leaves. The following *involution* provides the needed bijection.

Scan the leaves from left to right until you either encounter a leaf labeled 2*, in which case you make it into an internal vertex and make it give birth to two new leaves each labeled* 1*, or you encounter a leaf labeled* 1 *whose sister is also labeled* 1*, in which case you remove them both and make their mother a new leaf labeled* 2*.* □

The reason they needed (1) was to prove the following rather hairy identity

$$\sum_{\substack{m_1,\dots,m_t\geq 1\\ m_1+\dots+m_t=l}} (-1)^t \binom{l-m_1}{m_1-1}\binom{l-m_2}{m_2}\cdots\binom{l-m_t}{m_t} \;=\; (-1)^l C_{l-1} \quad . \tag{2}$$

Rather than using (1) we will present two direct, shorter proofs. In fact we will prove the more general identity, for $l \geq m$,

$$\sum_{\substack{m_1,\dots,m_t\geq 1\\ m_1+\dots+m_t=m}} (-1)^t \binom{l-m_1}{m_1-1}\binom{l-m_2}{m_2}\cdots\binom{l-m_t}{m_t} \;=\; (-1)^m C_{m-1} \quad . \tag{2'}$$

Identity (2) is the special case $l=m$ of (2′).

Calling the left side of (2′) $A(l,m)$, separating the case $t=1$ (that yields $-\binom{l-m}{m-1}$) and summing over m_t (let's call it k), we readily get the recurrence

$$A(l,m) = -\binom{l-m}{m-1} - \sum_{k=1}^{m-1} \binom{l-k}{k} A(l,m-k) \quad .$$

It would then follow by induction on m that $A(l,m)$ equals $(-1)^m C_{m-1}$, if the latter satisfies the same recurrence. But this is equivalent to the identity

$$\sum_{k=0}^{m} (-1)^k \binom{l-m+k}{m-k} C_k = \binom{l-m-1}{m} \quad . \tag{3}$$

Proof 1 (by SBE): Dividing by the right hand side, this is, in turn, equivalent to

$$\sum_{k=0}^{m} (-1)^k \frac{\binom{l-m+k}{m-k}}{\binom{l-m-1}{m}} C_k \;=\; 1 \quad .$$

Calling the left side $f(m)$, go into Maple and type

```
SumTools[Hypergeometric][ZeilbergerRecurrence]((-1)**k*binomial(2*k,k)/(k+1)*binomial(l-
m+k,m-k)/binomial(l-m-1,m),m,k,f,0..m);
```

and in one nano-second you would get that $f(m)$ satisfies the following recurrence

$$-(m+1)\, f\,(m) + (m+2)\, f\,(m+1) = 1 \quad ,$$

Since $f(1)=1$ (check!), it follows by induction that $f(m)=1$ for every m (and of course every $l \geq m$). $\square$

Proof 2 (by DZ): The left side of (3) without the $(-1)^k$ is the number of pairs (T,w) where T is a binary tree with $\leq m+1$ leaves and w is a word in $\{1,2\}$ of length $l-m-1$ longer than the number of leaves of T, whose sum is l. Hence the left side of (3) is the difference between the number of such creatures where T has an odd number of leaves and those that have an even number of leaves. The above involution that proved (1) (where the leaves of T are labeled by the corresponding prefix of w) is still valid, but now the survivors are the pairs $(.,1w)$ where . is the one-leaf tree, and w is a word in $\{1,2\}$ of length $l-m-1$ whose sum is $l-1$. $\square$

Thanks are due to Victor S. Miller for bringing [SW] to out attention. Also thanks to Lisa Sauermann for very insightful and useful comments on an earlier draft.

Reference

[SW] Lisa Sauermann and Yuval Wigderson, *Polynomials that vanish to high order on most of the hypercube*, arXiv:2010.00077v1 [math. C0] 30 Sep 2020

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Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger and arxiv.org

First written:**Nov. 11, 2020** . This version: **Nov. 15, 2020**.