Tutorial on Summation and #1 of DeNardo's assignment on Least Squares

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Reminder about simple average

If $X_1, \ldots, X_n$ are n numbers their **average** denoted by $\bar{X}$ is defined by

$$\bar{X} = \frac{X_1 + \ldots + X_n}{n} \quad ,$$

and in fancy **sigma notation**

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad .$$

An important consequence is that

$$\sum_{i=1}^{n} X_i = n\bar{X} \quad .$$

For the special case $n = 2$, we have

$$X_1 + X_2 = 2\bar{X} \quad ,$$

and for the special case $n = 3$, we have

$$X_1 + X_2 + X_3 = 3\bar{X} \quad .$$

Reminder on Summation

Given n numbers, $a_1, a_2, \ldots a_n$, their sum, $a_1 + a_2 + ... + a_n$ is abbreviated as

$$\sum_{i=1}^{n} a_i \quad .$$

There are simple rules on handling $\sum$ expressions. If c is any number, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right) \qquad (Rule1)$$

Indeed, when $n = 2$, for example, the left side is

$$ca_1 + ca_2$$

simplifying, we get that this is

$$c(a_1 + a_2)$$

which is the same as the right side for $n = 2$.

Again, for $n = 3$. The right side is

$$ca_1 + ca_2 + ca_3$$

simplifying, we get that this is

$$c(a_1 + a_2 + a_3)$$

which is the same as the right side for $n = 3$.

Another obvious rule (in fact a special case) is

$$\sum_{i=1}^{n} c = cn \qquad (Rule)$$

Since

$$\sum_{i=1}^{n} c = \sum_{i=1}^{n} c \cdot 1 = c\left(\sum_{i=1}^{n} 1\right) = cn \quad ,$$

since $\sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1$ (n times) and this equals n.

Now I will spell-out Problem 1 for $n = 2$ and $n = 3$, but you would have to do it for general n using $\sum$ notation.

When **n=2** we have

$$w_1 = \frac{X_1 - \bar{X}}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2} \quad , \quad w_2 = \frac{X_2 - \bar{X}}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2} \quad .$$

For the sake of simplicity let's call the denominator (that is some number, and it is the same for w_1 and w_2), D. In other words

$$D = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 \quad .$$

So abbreviating

$$w_1 = \frac{X_1 - \bar{X}}{D} \quad , \quad w_2 = \frac{X_2 - \bar{X}}{D} \quad .$$

When $n = 2$ the three assertions spell-out to be

$$w_1 + w_2 = 0 \qquad (Assertion1)$$

$$w_1 x_1 + w_2 x_2 = 1 \qquad (Assertion2)$$

$$w_1^2 + w_2^2 = \frac{1}{D} \qquad (Assertion3)$$

As for $(Assertion1)$

$$w_1 + w_2 = \frac{X_1 - \bar{X}}{D} + \frac{X_2 - \bar{X}}{D} = \frac{X_1 - \bar{X} + X_2 - \bar{X}}{D} = \frac{X_1 + X_2 - 2\bar{X}}{D} = \frac{0}{D} = 0 \quad ,$$

since, as we said above (for $n = 2$) $X_1 + X_2 = 2\bar{X}$, so $X_1 + X_2 - 2\bar{X} = 0$.

For ($Assertion2$), it is better to use x_i, (rather than X_i), so we have, from the definition

$$w_1 = \frac{x_1}{D} \quad , \quad w_2 = \frac{x_2}{D} \quad .$$

So

$$\sum_{i=1}^{2} x_i w_i = x_1 w_1 + x_2 w_2 = x_1 \frac{x_1}{D} + x_2 \frac{x_2}{D} = \frac{x_1^2}{D} + \frac{x_2^2}{D} = \frac{x_1^2 + x_2^2}{D} = 1 \quad ,$$

since **by definition**, $D = x_1^2 + x_2^2$.

For ($Assertion3$), it is also better to use x_i

$$\sum_{i=1}^{2} w_i^2 = w_1^2 + w_2^2 = \left(\frac{x_1}{D}\right)^2 + \left(\frac{x_2}{D}\right)^2 = \frac{x_1^2}{D^2} + \frac{x_2^2}{D^2} = \frac{x_1^2 + x_2^2}{D^2} \quad .$$

But, **by definition** (when $n = 2$), $D = x_1^2 + x_2^2$, so this is

$$\frac{D}{D^2} = \frac{1}{D} = \frac{1}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2} \quad .$$

This concludes the case $n = 2$.

Now let's do the same for $n = 3$.

$$w_1 = \frac{X_1 - \bar{X}}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + +(X_3 - \bar{X})^2} \quad ,$$

$$w_2 = \frac{X_2 - \bar{X}}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + +(X_3 - \bar{X})^2} \quad ,$$

$$w_3 = \frac{X_2 - \bar{X}}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + +(X_3 - \bar{X})^2} \quad ,$$

For the sake of simplicity let's call the denominator (that is some number, and it is the same for w_1, w_2 and w_3), D. In other words

$$D = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + +(X_3 - \bar{X})^2 \quad .$$

So abbreviating

$$w_1 = \frac{X_1 - \bar{X}}{D} \quad , \quad w_2 = \frac{X_2 - \bar{X}}{D} \quad , \quad w_3 = \frac{X_3 - \bar{X}}{D} \quad .$$

In terms of x_i rather than X_i, we have

$$w_1 = \frac{x_1}{D} \quad , \quad w_2 = \frac{x_2}{D} \quad , \quad w_3 = \frac{x_3}{D} \quad .$$

When $n = 3$ the three assertions spell-out to be

$$w_1 + w_2 + x_3 = 0 \qquad (Assertion1)$$

$$w_1 x_1 + w_2 x_2 + w_3 x_3 = 1 \qquad (Assertion2)$$

$$w_1^2 + w_2^2 + w_3^2 = \frac{1}{D} \qquad (Assertion3)$$

As for $(Assertion1)$

$$w_1 + w_2 + w_3 = \frac{X_1 - \bar{X}}{D} + \frac{X_2 - \bar{X}}{D} + \frac{X_3 - \bar{X}}{D} = \frac{X_1 + X_2 + X_3 - 3\bar{X}}{D} = \frac{0}{D} = 0 \quad ,$$

since, as we said above (for $n = 3$) $X_1 + X_2 + X_3 = 3\bar{X}$, so $X_1 + X_2 + X_3 - 3\bar{X} = 0$.

For $(Assertion2)$, it is better to use x_i, (rather than X_i), so we have, from the definition

$$w_1 = \frac{x_1}{D} \quad , \quad w_2 = \frac{x_2}{D} \quad , \quad w_3 = \frac{x_3}{D} \quad .$$

So

$$\sum_{i=1}^{3} x_i w_i = x_1 w_1 + x_2 w_2 + x_3 w_3 = x_1 \frac{x_1}{D} + x_2 \frac{x_2}{D} + x_3 \frac{x_3}{D} = \frac{x_1^2}{D} + \frac{x_2^2}{D} + \frac{x_3^2}{D} = \frac{x_1^2 + x_2^2 + x_3^2}{D} = 1 \quad ,$$

since **by definition** $D = x_1^2 + x_2^2 + x_3^2$.

For $(Assertion3)$, it is also better to use x_i

$$\sum_{i=1}^{3} w_i^2 = w_1^2 + w_2^2 + w_3^2 = \left(\frac{x_1}{D}\right)^2 + \left(\frac{x_2}{D}\right)^2 + \left(\frac{x_3}{D}\right)^2$$

$$= \frac{x_1^2}{D^2} + \frac{x_2^2}{D^2} + \frac{x_3^2}{D^2} = \frac{x_1^2 + x_2^2 + x_3^2}{D^2} \quad .$$

But, **by definition** (when $n = 3$), $D = x_1^2 + x_2^2 + x_3^2$, so this is

$$\frac{D}{D^2} = \frac{1}{D} = \frac{1}{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + (X_3 - \bar{X})^2} \quad .$$

Now it is **your job** to do it for general n using $\sum$ notation!

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