

**Dodgson's Determinant-Evaluation Rule Proved by  
TWO-TIMING MEN and WOMEN**

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*Bijections are where it's at —Herb Wilf*

*Dedicated to Master Bijectionist Herb Wilf, on finishing 13/24 of his life*

I will give a bijective proof of the Reverend Charles Lutwidge **Dodgson's Rule**([D]):

$$\det \left[ (a_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \right] \cdot \det \left[ (a_{i,j})_{\substack{2 \leq i \leq n-1 \\ 2 \leq j \leq n-1}} \right] = \\ \det \left[ (a_{i,j})_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1}} \right] \cdot \det \left[ (a_{i,j})_{\substack{2 \leq i \leq n \\ 2 \leq j \leq n}} \right] - \det \left[ (a_{i,j})_{\substack{1 \leq i \leq n-1 \\ 2 \leq j \leq n}} \right] \cdot \det \left[ (a_{i,j})_{\substack{2 \leq i \leq n \\ 1 \leq j \leq n-1}} \right] . \quad (\text{Alice})$$

Consider  $n$  men,  $1, 2, \dots, n$ , and  $n$  women  $1', 2' \dots, n'$ , each of whom is married to exactly one member of the opposite sex. For each of the  $n!$  possible (perfect) matchings  $\pi$ , let

$$\text{weight}(\pi) := \text{sign}(\pi) \prod_{i=1}^n a_{i,\pi(i)} \quad ,$$

where  $\text{sign}(\pi)$  is the sign of the corresponding permutation, and for  $i = 1, \dots, n$ , Mr.  $i$  is married to Ms.  $\pi(i)'$ .

Except for Mr. 1, Mr.  $n$ , Ms.  $1'$  and Ms.  $n'$  all the persons have affairs. Assume that each of the men in  $\{2, \dots, n-1\}$  has exactly one mistress amongst  $\{2', \dots, (n-1)'\}$  and each of the women in  $\{2', \dots, (n-1)'\}$  has exactly one lover amongst  $\{2, \dots, n-1\}$ <sup>2</sup>. For each of the  $(n-2)!$  possible (perfect) matchings  $\sigma$ , let

$$\text{weight}(\sigma) := \text{sign}(\sigma) \prod_{i=2}^{n-1} a_{i,\sigma(i)} \quad ,$$

where  $\text{sign}(\sigma)$  is the sign of the corresponding permutation, and for  $i = 2, \dots, n-1$ , Mr.  $i$  is the lover of Ms.  $\sigma(i)'$ .

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Supported in part by the NSF. Version of May 8, 1996. First Version: April 16, 1996. Submitted to the Wilf (Electronic) Festschrift.

Thanks are due to Bill Gosper for several corrections.

<sup>2</sup> Somewhat unrealistically, a man's wife may also be his mistress, and equivalently, a woman's husband may also be her lover.

Let  $A(n)$  be the set of all pairs  $[\pi, \sigma]$  as above, and let  $weight([\pi, \sigma]) := weight(\pi)weight(\sigma)$ . The left side of (*Alice*) is the sum of all the weights of the elements of  $A(n)$ .

Let  $B(n)$  be the set of pairs  $[\pi, \sigma]$ , where now  $n$  and  $n'$  are unmarried but have affairs, i.e.  $\pi$  is a matching of  $\{1, \dots, n-1\}$  to  $\{1', \dots, (n-1)'\}$ , and  $\sigma$  is a matching of  $\{2, \dots, n\}$  to  $\{2', \dots, n'\}$ , and define the weight similarly.

Let  $C(n)$  be the set of pairs  $[\pi, \sigma]$ , where now  $n$  and  $1'$  are unmarried and  $1$  and  $n'$  don't have affairs. i.e.  $\pi$  is a matching of  $\{1, \dots, n-1\}$  to  $\{2', \dots, n'\}$ , and  $\sigma$  is a matching of  $\{2, \dots, n\}$  to  $\{1', \dots, (n-1)'\}$ , and now define  $weight([\pi, \sigma]) := -weight(\pi)weight(\sigma)$ .

The right side of (*Alice*) is the sum of all the weights of the elements of  $B(n) \cup C(n)$ .

Define a mapping

$$T : A(n) \rightarrow B(n) \cup C(n) \quad ,$$

as follows. Given  $[\pi, \sigma] \in A(n)$ , define an alternating sequence of men and women:  $m_1 := n, w_1, m_2, w_2, \dots, m_r, w_r = 1'$  or  $n'$ , such that  $w_i := \text{wife of}(m_i)$ , and  $m_{i+1} := \text{lover of}(w_i)$ . This sequence terminates, for some  $r$ , at either  $w_r = 1'$ , or  $w_r = n'$ , since then  $m_{r+1}$  is undefined, as  $1'$  and  $n'$  are lovers-less women. To perform  $T$ , change the relationships  $(m_1, w_1), (m_2, w_2), \dots, (m_r, w_r)$  from marriages to affairs (i.e. Mr.  $m_i$  and Ms.  $w_i$  get divorced and become lovers,  $i = 1, \dots, r$ ), and change the relationships  $(m_2, w_1), (m_3, w_2), \dots, (m_r, w_{r-1})$  from affairs to marriages. If  $w_r = 1'$  then  $T([\pi, \sigma]) \in C(n)$ , while if  $w_r = n'$  then  $T([\pi, \sigma]) \in B(n)$ .

The mapping  $T$  is weight-preserving. Except for the sign, this is obvious, since all the relationships have been preserved, only the nature of some of them changed. I leave it as a pleasant exercise to verify that also the sign is preserved.

It is obvious that  $T : A(n) \rightarrow B(n) \cup C(n)$  is one-to-one. If it were onto, we would be done. Since it is not, we need one more paragraph.

Call a member of  $B(n) \cup C(n)$  *bad* if it is not in  $T(A(n))$ . I claim that the sum of all the weights of the bad members of  $B(n) \cup C(n)$  is zero. This follows from the fact that there is a natural bijection  $S$ , easily constructed by the readers, between the bad members of  $C(n)$  and those of  $B(n)$ , such that  $weight(S([\pi, \sigma])) = -weight([\pi, \sigma])$ . Hence the weights of the bad members of  $B(n)$  and  $C(n)$  cancel each other in pairs, contributing a total of zero to the right side of (*Alice*).  $\square$

A small Maple package, *alice*, containing programs implementing the mapping  $T$ , its inverse, and the mapping  $S$  from the bad members of  $C(n)$  to those of  $B(n)$ , is available from my Home Page <http://www.math.temple.edu/~zeilberg>.

## Reference

[D] C.L. Dodgson, *Condensation of Determinants*, Proceedings of the Royal Society of London **15**(1866), 150-155.