Human and automated approaches for finite trigonometric sums

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Abstract

We show that identities involving trigonometric sums recently proved by Harshitha, Vasuki and Yathirajsharma, using Ramanujan's theory of theta functions, were either already in the literature or can be proved easily by adapting results that can be found in the literature. Also we prove two conjectures given in that paper. Finally we give an automated approach for proving such trigonometric identities.

Identities between quantities that are not clearly and immediately related are always fascinating. In particular when their proofs use subtle or complicated arguments (think of the Basel Problem, and the solution by Euler, $\sum 1/n^2 = \pi^2/6$). But it also happens that, after a "complicated" proof, people try to find a "direct" or an "elementary" proof: in some cases this is not (not yet?) possible (think of the Fermat-Wiles theorem). Trigonometric identities, *a priori* unexpected, can also be found in the literature. We will just cite two such identities, one due to Gauss (see, e.g., [9]):

$$\cos\left(\frac{2\pi}{17}\right) = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}}{16}$$

the other is a consequence of the Ramanujan theta function identity (combine the first two equalities of Lemma 3.8 in [4], or see [2, p. 184]; also see Eq. (1.31) of Corollary 7 in [15], and Eq. (1.11) in [11]):

$$\frac{\sin^2\left(\frac{3\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} - \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)} + \frac{\sin^2\left(\frac{\pi}{7}\right)}{\sin\left(\frac{3\pi}{7}\right)} = 0.$$

Much more modestly we stumbled upon the recent paper [11] published in the Ramanujan journal, where the authors give an interesting way to obtain a closed form for several trigonometric sums by using Ramanujan's theory of theta functions. In particular Theorem 1.1 in that paper gives six formulas about which the authors write that these six identities *seem to be new*. Of course it is very interesting to obtain these equalities through theta functions, but we were interested to search whether these formulas previously appeared in the literature. Having found a vast set of papers on the subject of identities involving finite trigonometric sums, we were lucky enough to discover either references in which some of these identities were already be proved, or known results from which the other identities can be easily deduced. Here we describe or adapt these results. We also confirm two conjectures given by the authors of [11] at the end of their paper.

Finally we show how to give automated proofs for trigonometric identities of this kind.

1 The six identities of Harshitha, Vasuki and Yathirajsharma

First we recall the six trigonometric identities given in [11, Theorem 1.1].

1. *If k is an odd natural number, then*

$$\sum_{j=1}^{\frac{k-1}{2}} (-1)^{j-1} \sin\left(\frac{(2j-1)\pi}{2k}\right) = \frac{(-1)^{\frac{k-3}{2}}}{2} \quad \text{(here one should suppose that } k \geq 3\text{)} \tag{1}$$

and

$$\sum_{j=1}^{\frac{k-1}{2}} (-1)^{j-1} \csc\left(\frac{(2j-1)\pi}{2k}\right) = \frac{k+(-1)^{\frac{k+1}{2}}}{2}. \tag{2}$$

2. *If n is an odd natural number and $j = 2p$ is an even positive integer such that* $\gcd(j,n) = 1$, *then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2-1}{3}. \tag{3}$$

3. *If n is an even number and $j \equiv 2 \pmod 4$, such that* $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$, *then*

$$\sum_{k=0}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2}{4}. \tag{4}$$

4. *If n is an even number and $j \equiv 2 \pmod 4$, such that* $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$, *then*

$$\sum_{k=0}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)k\pi}{n}\right)\sin\left(\frac{(j-1)k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)\sin^2\left(\frac{jk\pi}{n}\right)} = \frac{n^2-4}{12}. \tag{5}$$

5. *If n is an odd natural number and j is a positive integer such that* $\gcd(j,n) = 1$, *then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} = 0. \tag{6}$$

2 Some trigonometric identities found in the literature

Here we give three propositions that state results we found in the literature.

Proposition 1 ([16], last displayed formula)

$$\sum_{j=1}^{n}(-1)^{j+1}\sin\left(\frac{(2j-1)\pi}{4n+2}\right)=\frac{(-1)^{n+1}}{2}.$$

Remark 2 Several sums of the same kind appear in the literature, where sin can be replaced with cos or a power of cos or sin, for example a companion formula of the formula above is given in [16], namely, for $n \geq 1$,

$$\sum_{j=1}^{n}(-1)^{j+1}\cos\left(\frac{j\pi}{2n+1}\right)=\frac{1}{2}.$$

Also the relations

$$\sum_{k=1}^{n}(-1)^{k+1}\cos^2\left(\frac{k\pi}{2n+2}\right)=\frac{1}{2} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^{k+1}\sin^2\left(\frac{k\pi}{2n+2}\right)=\frac{(-1)^{n+1}}{2}$$

can be found in [20]. The same formulas can be found in [1, Theorem 3.4, p. 218].

Note that there is a straightfoward proof of the relation in Proposition 1 as well as of the three relations above, by using the relation (see, e.g., [8, Example 7, p. 405])

$$\cos(\alpha)+\cos(\alpha+\beta)+\cdots+\cos(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{1}{2}\beta n\right)}{\sin\left(\frac{1}{2}\beta\right)}\cos\left(\alpha+\frac{1}{2}\beta(n-1)\right)$$

and replacing the possible occurrences of $\cos^2(x)$ and $\sin^2 x$ with $(1\pm\cos 2x)/2$.

Proposition 3 ([12], Relation 3.16 on p. 817)

$$\sum_{j=0}^{n-1}(-1)^{j}\csc\left(\frac{(2j+1)\pi}{4n+2}\right)=\frac{2n+1-(-1)^n}{2}.$$

Now we give the last proposition of this section that we discovered in a 1908 book by Bromwich.

Proposition 4 ([5], p. 183; also see p. 187)

$$\begin{cases}
\text{(I)} & \displaystyle\sum_{k=1}^{\frac{n-1}{2}}\frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} = \frac{n^2-1}{6} & \textit{if } n \textit{ is odd;}\\[2ex]
\text{(II)} & \displaystyle\sum_{k=1}^{\frac{n}{2}-1}\frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} = \frac{n^2-4}{6} & \textit{if } n \textit{ is even;}\\[2ex]
\text{(III)} & \displaystyle\sum_{k=0}^{\frac{n-3}{2}}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = \frac{n^2-1}{2} & \textit{if } n \textit{ is odd;}\\[2ex]
\text{(IV)} & \displaystyle\sum_{k=0}^{\frac{n}{2}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = \frac{n^2}{2} & \textit{if } n \textit{ is even.}
\end{cases}$$

Remark 5 One can find in [7, Identity (23)] the identity

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} = \frac{n^2-1}{3}.$$

The case where n is odd is an exercise in Bromwich's book ([5, Ex. 4, p. 188]). Also see [10] where it is stated that such identities were conjectured while computing low-temperature series for a Z_n-symmetric Hamiltonian in statistical mechanics [14]; (also see [13] and the references therein for much more on this and similar identities). Actually this identity can be easily deduced from (I) and (II) in Proposition 4 above (cut the sum into $\sum_{k=1}^{(n-1)/2} + \sum_{(n+1)/2}^{n-1}$ if n is odd, and into $\sum_{k=1}^{n/2} + \sum_{(n/2)+1}^{n-1}$ if n is even, and make the change of index $\ell = n - k$ in each of the second sums).

Remark 6 One can find in [6] (also see the other references given there) the identity

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = n^2.$$

Note that this equality is given in [11] under the slightly disguised form

$$\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} = n^2$$

as a corollary of the identity

$$\sum_{k=1}^{n-1} \cot^2\left(\frac{k\pi}{n}\right) = \frac{(n-1)(n-2)}{3} \quad \text{for any } n > 0.$$

The authors of [11] cite a similar derivation in [3, Corollary 2.4]. It is interesting to note that the "disguised" equality above is already given (actually only for n even) by Riesz in 1914 in [18, 19] where it is used to prove a theorem of Bernstein. Actually the identity at the beginning of this Remark 6 can be easily deduced from (III) and (IV) in Proposition 4 above, as was done in Remark 5.

Remark 7 Note that the identities (I, II, III, IV) in Proposition 4 are, up to notation, respectively the identities (3.40, 3.28, 3.14, 3.4) in the paper [12], where they are also generalized.

We end this section with an easy lemma.

Lemma 8 *The following equalities hold.*

(i) Let n be an odd integer and p an integer such that $\gcd(p, n) = 1$. *Then*

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{\ell=1}^{n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \frac{1}{2}\sum_{r=1}^{n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{n^2-1}{6}.$$

(ii) Let n be an even integer and p an odd integer, such that $\gcd(p, n/2) = 1$. *Then*

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{\ell=1}^{n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{r=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{n^2}{4}.$$

Proof. (i) For n odd and $\gcd(p, n) = 1$ we have

$$\begin{aligned}
\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{\substack{1\le\ell\le n-2\\ \ell \text{ odd}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} \\
&= \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{\substack{1\le\ell\le n-1\\ \ell \text{ even}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} \\
&= \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.
\end{aligned}$$

But

$$\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \sum_{\frac{n+1}{2}\le s\le n-1} \frac{1}{\sin^2\left(\frac{2ps\pi}{n}\right)} \quad \text{(change of index } s = n - r\text{)}.$$

Hence

$$\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} = \frac{1}{2}\left(\sum_{1\le r\le \frac{n-1}{2}} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)} + \sum_{\frac{n+1}{2}\le s\le n-1} \frac{1}{\sin^2\left(\frac{2ps\pi}{n}\right)}\right) = \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.$$

Thus, finally

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.$$

Now we claim that the first sum on the right hand side only depends on the value of $p\ell$ modulo n and that $p\ell$ takes exactly once all the values modulo n except the value 0 since $\gcd(p, n) = 1$, thus this sum is equal to the same sum where p is replaced with 1. Similarly the second sum only depends on the value of $2pr$ modulo n and $2pr$ takes exactly once all the values modulo n except the value 0 since $\gcd(2p, n) = 1$, thus the sum is equal to the same sum where $2p$ is replaced with 1. Thus

$$\sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \sum_{1\le\ell\le n-1} \frac{1}{\sin^2\left(\frac{\ell\pi}{n}\right)} - \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{r\pi}{n}\right)} = \frac{1}{2}\sum_{1\le r\le n-1} \frac{1}{\sin^2\left(\frac{r\pi}{n}\right)}.$$

This last sum is equal to $\frac{n^2-1}{6}$ from Remark 5, where it was indicated that the first identity there is a consequence of Proposition 4 (I).

(*ii*) For n even, p odd and $\gcd(p, n/2) = 1$ we have

$$\begin{aligned}
\sum_{k=0}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ odd}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} \\
&= \sum_{1\leq \ell \leq n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ even}}} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} \\
&= \sum_{1\leq \ell \leq n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} - \sum_{1\leq r \leq \frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{2pr\pi}{n}\right)}.
\end{aligned}$$

As seen above the first sum on the right hand side does not depend on p. It is equal to $\frac{n^2-1}{3}$. Similarly, defining $m := n/2$, the second sum can be written

$$\sum_{1\leq r\leq m-1} \frac{1}{\sin^2\left(\frac{pr\pi}{m}\right)}$$

where $\gcd(p, m) = 1$. It is thus equal to $\frac{m^2-1}{3}$. Hence, finally

$$\sum_{1\leq k\leq \frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} = \frac{n^2}{4}.$$

Remark 9 In the course of the proofs above, we proved, for $\gcd(p, n) = 1$, the identity

$$\sum_{1\leq \ell\leq n-1} \frac{1}{\sin^2\left(\frac{p\ell\pi}{n}\right)} = \frac{n^2-1}{3}$$

(compare with Remark 5).

3 Identities (1) to (6) revisited

We begin this section with a straightforward trigonometric formula, namely

$$(\#) \qquad \frac{\sin x \sin y}{\sin^2(\frac{x+y}{2})\sin^2(\frac{x-y}{2})} = \frac{1}{\sin^2(\frac{x-y}{2})} - \frac{1}{\sin^2(\frac{x+y}{2})}$$

(write: $2\sin x \sin y = \cos(x-y) - \cos(x+y)$ and use $\cos z = 1 - 2\sin^2(z/2)$).

Now we revisit Identities (1) to (6), to show that they were either already in the literature or easy consequences of identities in the literature.

- Identity (1)

  See Proposition 1.

- Identity (2)

  See Proposition 3.

- Identity (3)

  We have for n odd, $j = 2p$ and $\gcd(j, n) = 1$, successively using (#), Proposition 4 (III), and Lemma 8 (ii),

$$\begin{aligned}\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} - \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} \\ &= \frac{n^2-1}{2} - \frac{n^2-1}{6} = \frac{n^2-1}{3}.\end{aligned}$$

- Identity (4)

  We have for n even, $j = 2p$, $j \equiv 2 \pmod 4$, and $\gcd(\frac{j}{2}, \frac{n}{2}) = 1$ successively using (#), Proposition 4 (IV), and Lemma 8 (i)

$$\begin{aligned}\sum_{k=1}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{2n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{2n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{p(2k+1)\pi}{n}\right)} \\ &= \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}.\end{aligned}$$

- Identity (5)

  We have for n even, $j = 2p$, $j \equiv 2 \pmod 4$, and $\gcd(j/2, n/2) = 1$, successively using (#), Proposition 4 (II), and the identity in Remark 9,

$$\begin{aligned}\sum_{k=1}^{\frac{n}{2}-1} \frac{\sin\left(\frac{(j+1)k\pi}{n}\right)\sin\left(\frac{(j-1)k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)\sin^2\left(\frac{jk\pi}{n}\right)} &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{jk\pi}{n}\right)} \\ &= \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\frac{(j/2)k\pi}{n/2}\right)} \\ &= \frac{n^2-4}{6} - \frac{(\frac{n}{2})^2-1}{3} = \frac{n^2-4}{12}.\end{aligned}$$

- Identity (6)

  We have for n odd and j such that $\gcd(j, n) = 1$, successively using (#), and Lemma 8(i),

$$\begin{aligned}\sum_{k=0}^{\frac{n-3}{2}} \frac{\sin\left(\frac{(j+1)(2k+1)\pi}{n}\right)\sin\left(\frac{(j-1)(2k+1)\pi}{n}\right)}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} &= \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{n}\right)} - \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{\sin^2\left(\frac{j(2k+1)\pi}{n}\right)} \\ &= 0.\end{aligned}$$

4 Confirming two conjectures of Harshitha, Vasuki, Yathirajsharma

At the end of their paper [11], the authors give two conjectures about which they wrote: *We feel that these two can also be tackled with Ramanujan's theory although we were unable to do so.* First we recall these conjectures.

Conjecture [11] *If k is any positive integer, then*

$$\lim_{k\to\infty}\left\{\sum_{j=0}^{2k-1}(-1)^j\sin^2\left(\frac{(2j+1)\pi}{8k+2}\right)\right\}=-\frac{1}{2}$$

and

$$\lim_{k\to\infty}\left\{\sum_{j=0}^{2k}(-1)^j\sin^2\left(\frac{(2j+1)\pi}{8k+6}\right)\right\}=\frac{1}{2}\cdot$$

Here we give a simple proposition (but without Ramanujan's theory of theta functions) which easily implies both conjectures above.

Proposition 10 *The following equalities hold.*

$$\sum_{j=0}^{2k-1}(-1)^j\sin^2\left(\frac{(2j+1)\pi}{8k+2}\right)=\frac{-\sin^2\left(\dfrac{2k\pi}{4k+1}\right)}{2\cos\left(\dfrac{\pi}{4k+1}\right)}$$

and

$$\sum_{j=0}^{2k}(-1)^j\sin^2\left(\frac{(2j+1)\pi}{8k+6}\right)=\frac{1}{2}-\frac{\cos^2\left(\dfrac{(2k+1)\pi}{4k+3}\right)}{2\cos\left(\dfrac{\pi}{4k+3}\right)}\cdot$$

As a consequence the conjectures above hold.

Proof. The proof is left to the reader who can use the relations $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$, $(-1)^t\cos y=\cos(t\pi+y)$, and, as previously, the identity [8, Example 7, p. 405],

$$\cos(\alpha)+\cos(\alpha+\beta)+\cdots+\cos(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{1}{2}\beta n\right)}{\sin\left(\frac{1}{2}\beta\right)}\cos\left(\alpha+\frac{1}{2}\beta(n-1)\right).\qquad\square$$

5 Automated proofs of the above and similar identities

Two hints seemed to be in favor of possible automated proofs (or "shaloshable proofs", see [21]; also see [17]) of the above and similar trigonometric identities. The (very vague) first hint is the occurrence of a hypergeometric series in [11, Theorem 2.1]. The second hint is the following identity that can be found, e.g., in [5, p. 184]

$$\sin(n\theta)=2^{n-1}\prod_{r=0}^{n-1}\sin\left(\theta+\frac{r\pi}{n}\right)\cdot\tag{7}$$

This identity can be used to prove a bunch of identities. For example, taking its logarithmic derivative, then differentiating, yields

$$\frac{n^2}{\sin^2(n\theta)}-\frac{1}{\sin^2(\theta)}=\sum_{r=1}^{n-1}\frac{1}{\sin^2\left(\theta+\frac{r\pi}{n}\right)}\cdot$$

Now, letting θ tend to 0, one obtains the identity of Remark 5. But one way to obtain Identity (7) is to use Cebyshev polynomials.

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