A STEMBRIDGE-STANTON STYLE PROOF OF THE HABSIEGER-KADELL q-MORRIS IDENTITY

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0. NOMENCLATURE

q and $x = (x_1, ..., x_n)$ are commuting indeterminates. If $\alpha = (\alpha_1, ..., \alpha_n)$ is any vector of integers , then x^α stands for $x^{\alpha_1} ... x^{\alpha_n}$. For example if α=(1,-2,5), then $x^\alpha = x_1 x_2^{-2} x_3^5$.

A *Laurent polynomial* is a finite linear combination of monomials x^α, where the αs have integer components. All our Laurent polynomials will be with integer coefficients.

"C.T." stands for "the constant term of", with respect to $x = (x_1, ..., x_n)$. For example C.T.(1-qx)(1-q/x)= 1+q.

The symmetric group S_n acts on vectors of integers by permuting the coordinates, for example 321(-1,2,1)=(1,2,-1). A permutation π acts on monomials x^γ by $\pi(x^\gamma) = x^{\pi(\gamma)}$, and by linearity on any Laurent polynomial. For example,

$$(321)[x_1^{-1} x_2^2 x_3 + 4 + x_1^2 x_2^3 x_3^{-5}] = x_1 x_2^2 x_3^{-1} + 4 + x_1^{-5} x_2^3 x_1^2.$$

A Laurent polynomial P in $x = (x_1, ..., x_n)$ is *symmetric* if $\pi(P) = P$ for all permutations π, and is *antisymmetric* if $\pi(P) = (sgn\ \pi)P$ for every permutation π.

$(y; Q)_a$, the "q-analog of $(1-y)^a$ to base Q " is defined by

$$(y; Q)_a = (1-y)(1-Qy)...(1-Q^{a-1}y),$$

and if the base Q is q then we often abbreviate $(y; q)_a$ to $(y)_a$.

A vector of integers α is a *bad guy* if it has two or more identical components, otherwise it is a *good guy*. For example $(1, 3, 1)$ and $(-1, -1, -1)$ are bad guys while $(1, -1, 0)$ and $(2, 1, 8)$ are good guys.

δ is the vector $(0, 1, ..., n-1)$ and $\bar{\delta}$ is its reverse: $\bar{\delta} = (n-1, ..., 0)$.

Throughout this paper $t = q^a$, $s = q^b$, $u = q^c$.

1. THE HABSIEGER-KADELL q-MORRIS IDENTITY

Let, for a,b,c,n nonnegative integers,

$$F'_{a,b,c}{}^{(n)}(x) = \prod_{i=1}^{n} (x_i)_b (q/x_i)_c \prod_{1 \le i < j \le n} (x_i/x_j)_a (qx_j/x_i)_a \tag{1.1}$$

$$H'_{a,b,c}{}^{(n)} = C.T. F'_{a,b,c}{}^{(n)}. \tag{1.2}$$

$$R'_{a,b,c}{}^{(n)} = \prod_{j=0}^{n-1} \frac{(q)_{b+c+ja}(q)_{(j+1)a}}{(q)_{b+ja}(q)_{c+ja}(q)_a} \tag{1.3}$$

In this paper I give a new proof of

THEOREM'(The Habsieger-Kadell q-Morris identity)

$$H'_{a,b,c}{}^{(n)} = R'_{a,b,c}{}^{(n)}. \tag{1.4}$$

This result was conjectured by Morris[Mo] who proved the q=1 case. It was recently proved independently by Habsieger[H] and Kadell[K]. Both Habsieger and Kadell first proved a q-analog of Selberg's integral that was conjectured by Askey[As] and then deduced from it the q-Morris identity.

The q-Morris identity is a generalization of the so-called "A cases of Macdonald's root system conjecture"([Ma]) , also known as " the equal parameter case of the Zeilberger-Bressoud q-Dyson theorem" . The general q-Dyson theorem was proved in [Z-B]. Indeed substituting b=c=0 in the q-Morris identity (1.4) gives the equal parameter case of q-Dyson.

John Stembridge[Ste], standing on the shoulders of Dennis Stanton[Sta], has recently come up with a short, elegant and elementary proof of the equal-parameter case of q-Dyson. In this paper I adapt Stembridge's proof to give a relatively short, elegant and *elementary* proof of the q-Morris identity.

The word "elementary" has at least two meanings. The first one is the colloquial "Holmesian" one that means "easy". The second one is the technical-philosophical "Kroneckerian" meaning of only using finite algebraic operations on integers. The present proof is elementary in both senses. The statement of the q-Morris identity (1.4) is completely elementary and God-created and it was disturbing that so far one had to resort to such artificial man-made analytical notions as limits and q-integration to prove it.

2. AN EQUIVALENT IDENTITY AND THE ROLE OF ANTISYMMETRY

It turns out that instead of $F'_{a,b,c}{}^{(n)}$ of (1.1) it is much easier to consider

$$F_{a,b,c}{}^{(n)}(x) = \prod_{i=1}^{n} (x_i)_b (q/x_i)_c \prod_{1 \le i < j \le n} (x_i/x_j)_a (qx_j/x_i)_{a-1} \tag{2.1}$$

and to try and evaluate

$$H_{a,b,c}{}^{(n)} = C.T. F_{a,b,c}{}^{(n)}, \tag{2.2}$$

that will turn out to be equal to

$$R_{a,b,c}{}^{(n)} = \prod_{j=0}^{n-1} \frac{(q)_{b+c+ja}(q)_{(j+1)a-1}}{(q)_{b+ja}(q)_{c+ja}(q)_{a-1}} \tag{2.3}$$

We will actually prove instead of the original statement (1.4) of the q-Morris identity the identity (2.4) below that turns out to be equivalent to it.

THEOREM

$$H_{a,b,c}{}^{(n)} = R_{a,b,c}{}^{(n)} \tag{2.4}$$

The reason that $F_{a,b,c}{}^{(n)}(x)$ is more congenial than $F'_{a,b,c}{}^{(n)}(x)$ is that the former is almost anti-symmetric. Indeed, peeling off the first layer of $(x_i/x_j)_a$ yields

$$F_{a,b,c}{}^{(n)}(x) = \prod_{1\le i<j\le n} (1 - x_i/x_j) \prod_{i=1}^{n} (x_i)_b (q/x_i)_c \prod_{1\le i<j\le n} (qx_i/x_j)_{a-1} (qx_j/x_i)_{a-1} \tag{2.5}$$

$$= x_2^{-1} x_3^{-2} \ldots x_n^{-(n-1)} \cdot \prod_{1\le i<j\le n} (x_j - x_i) \cdot (something \quad symmetric).$$

Define δ = (0,1,2,..., n-1), and

$$G_{a,b,c}{}^{(n)}(x) = x^{\delta} F_{a,b,c}{}^{(n)}, \tag{2.6}$$

then it follows from (2.5) that $G_{a,b,c}{}^{(n)}$ is an antisymmetric Laurent polynomial. In terms of $G_{a,b,c}{}^{(n)}$, the quantity of interest $H_{a,b,c}{}^{(n)}$ is expressed as

$$H_{a,b,c}{}^{(n)} = C.T.(x^{-\delta} G_{a,b,c}{}^{(n)}). \tag{2.7}$$

The proof of the equivalence of the original q-Morris identity (1.4) and its variant (2.4) is a pleasant exercise in antisymmetry. We will not give it here since the proof in section 4 of [Ste] passes verbatim (see also section 3 of [Z]).

The reason antisymmetry is so important is the following

CRUCIAL LEMMA

Let $G = G(x) = G(x_1, ..., x_n)$ be an antisymmetric Laurent polynomial.

(i) For any vector of integers γ and any permutation π we have

$$C.T.[x^{\pi(\gamma)} G] = sgn\pi C.T.[x^{\gamma} G].$$

(ii) If γ= $(\gamma_1, ..., \gamma_n)$ is a bad guy (i.e. there are i and j , $1 \le i < j \le n$, such that $\gamma_i = \gamma_j$), then $C.T.[x^{\gamma} G] = 0$

PROOF: (i) follows straight from the definitions of antisymmetry and the action of a permutation on a monomial, while (ii) follows from (i) by using the transposition (ij) whose sign is -1.

3. INDUCTION ON n

Of course $H^{(0)}_{a,b,c} \equiv 1$ and $R^{(0)}_{a,b,c} \equiv 1$, so (2.4) is true for n=0 and we have a basis to start induction on n. From the definition (2.1) it follows that

$$F^{(n+1)}_{a,0,0}(x_1, ..., x_{n+1}) = F^{(n)}_{a,a,a-1}(x_1/x_{n+1}, ..., x_n/x_{n+1}) \tag{3.1}$$

Thus taking the constant term yields

$$H^{(n+1)}_{a,0,0} = H^{(n)}_{a,a,a-1} \tag{3.2}$$

From the definition (2.3) of $R^{(n)}_{a,b,c}$ we have

$$R^{(n+1)}_{a,0,0} = R^{(n)}_{a,a,a-1}. \tag{3.3}$$

So if we knew that (2.4) was true for n and *all* a,b,c then by plugging b=a,c=a-1 we would have that it is true for n+1 with b=c=0. This will take care of climbing up the n induction ladder. Now we have to show that for a fixed n, the truth of (2.4) for b=c=0 implies its truth for all b,c. So it seems that we have to climb first the c induction ladder: showing the truth of (2.4) for b=0 and all c, and then the b ladder :showing that (2.4) for b=0 implies it for all b. Luckily we get the first ascent *gratis*. Indeed, since

$$\prod_{1 \le i < j \le n} (x_i/x_j)_a (qx_j/x_i)_{a-1}$$

is homogeneous, we obviously have

$$H^{(n)}_{a,0,c} = C.T. F^{(n)}_{a,0,c} = C.T. F^{(n)}_{a,0,0} = H^{(n)}_{a,0,0}.$$

Also from the definition (2.3) we have

$$R^{(n)}_{a,0,c} = R^{(n)}_{a,0,0}.$$

So we know that if (2.4) is true for b=c=0 then it is true for b=0 and all c. It remains to climb the b induction ladder.

4. INDUCTION ON b

(2.4) would follow by induction on b once we show that

$$\frac{H^{(n)}_{a,b+1,c}}{H^{(n)}_{a,b,c}} = \frac{R^{(n)}_{a,b+1,c}}{R^{(n)}_{a,b,c}}. \tag{4.1}$$

A routine calculation using the definition (2.3) shows that ($t = q^a$)

$$\frac{R_{a,b+1,c}}{R_{a,b,c}} = \prod_{j=0}^{n-1} \frac{(q)_{b+c+ja+1}(q)_{b+ja}}{(q)_{b+c+ja}(q)_{b+ja+1}} = \prod_{j=0}^{n-1} \frac{(1-q^{b+c+ja+1})}{(1-q^{b+ja+1})} = \frac{(q^{b+c+1};t)_n}{(q^{b+1};t)_n} \quad . \tag{4.2}$$

Now by the definitions (2.1) (2.2) , and by peeling off the last layer out of $(x_i)_{b+1}$,

$$H_{a,b+1,c} = C.T.F^{(n)}_{a,b+1,c} = C.T. \prod_{i=1}^{n} (1-q^b x_i) F^{(n)}_{a,b,c} = C.T.\{x^{-\delta} \prod_{i=1}^{n} (1-q^b x_i) G_{a,b,c}{}^{(n)}\}$$

Now let $s = q^b$ and by expanding the product we get

$$H^{(n)}_{a,b+1,c} = \sum_{\beta} (-s)^{|\beta|} C.T.[x^{\beta-\delta} G^{(n)}_{a,b,c}] \tag{4.3}$$

where the sum is over all (0-1) vectors $\beta = (\beta_1, ..., \beta_n)$, and $|\beta| = \beta_1 + ... + \beta_n =$ (the number of ones in β).

Now comes the gory Stembridge-Stanton *massacre of the bad guys*. The only way $\beta - \delta$ can be a good guy is if β has the form (1,...1,0,...,0), where for some r between 0 and n there are r 1's followed by n-r 0's. The reason is, of course, that if β had a zero followed by a one, say in the i and i+1 places: $\beta_i = 0, \beta_{i+1} = 1$ then the i and i+1 components of $\beta - \delta$ are going to be equal to each other. By the *crucial lemma* (ii) the terms in (4.3) that correspond to bad guys vanish and (4.3) becomes

$$H^{(n)}_{a,b+1,c} = \sum_{r=0}^{n} (-s)^r C.T.[x_1...x_r \cdot x^{-\delta} G^{(n)}_{a,b,c}] \tag{4.4}$$

The term corresponding to r=0 in the above sum is nothing but $C.T.[x^{-\delta} G^{(n)}_{a,b,c}]$ alias $H^{(n)}_{a,b,c}$. We have thus expressed $H^{(n)}_{a,b+1,c}$ in terms of $H^{(n)}_{a,b,c}$ *and* (unfortunately) some of its "buddies". We would be done if we will be able to express all the terms that feature in (4.4) in terms of $H^{(n)}_{a,b,c}$. Luckily it is indeed possible and in the next section we will prove (set $s = q^b$, $t = q^a$, $u = q^c$)

$$C.T.[x_1...x_r \cdot x^{-\delta} G^{(n)}_{a,b,c}] = (-q)^r \frac{(t;t)_n (u;t)_r (qs;t)_{n-r}}{(t;t)_r (t;t)_{n-r} (qs;t)_n} H^{(n)}_{a,b,c} \tag{4.5}$$

Substituting in (4.4) we get

$$\frac{H^{(n)}_{a,b+1,c}}{H^{(n)}_{a,b,c}} = \sum_{r=0}^{n} (qs)^r \frac{(t;t)_n (u;t)_r (qs;t)_{n-r}}{(t;t)_r (t;t)_{n-r} (qs;t)_n} \tag{4.6}$$

In order to conclude the proof of (4.1) (modulo (4.5)) we must show that the right hand sides of (4.6) and (4.2) are the same, i.e. we have to show (as before we set $t = q^a, s = q^b, u = q^c$)

$$\frac{(qsu;t)_n}{(qs;t)_n} = \sum_{r=0}^{n} (qs)^r \frac{(t;t)_n (u;t)_r (qs;t)_{n-r}}{(t;t)_r (t;t)_{n-r} (qs;t)_n} \tag{4.7}$$

But (4.7) follows immediately by setting X=qs, Y=u, in the following identity (4.8), taking the base to be t instead of the customary q (i.e. $()_a = (;t)_a$)

SIMPLE LEMMA (A variant of q-Vandermonde)

$$(XY)_n = \sum_{r=0}^{n} X^r \frac{(t)_n}{(t)_r (t)_{n-r}} (Y)_r (X)_{n-r} \tag{4.8}$$

PROOF OF THE SIMPLE LEMMA

Cauchy's famous q-analog of the binomial theorem (e.g. [An] p.10, (2.9)) says

$$\frac{(az;t)_\infty}{(z;t)_\infty} = \sum \frac{(a;t)_n}{(t;t)_n} z^n \tag{4.9}$$

(Incidentally, the " $|z| < 1, |t| < 1$" that is added as a "condition of validity " in [An] is completely superfluous, at least in *my* book).

Of course

$$\frac{(zXY;t)_\infty}{(z;t)_\infty} = \frac{(zXY;t)_\infty}{(zX;t)_\infty} \frac{(zX;t)_\infty}{(z;t)_\infty} \tag{4.10}$$

Now, using (4.9), we expand each of the three ratios in (4.10) as formal power series in z, and compare coefficients of z^n, which yields the desired identity (4.8).

We have thus completed the proof of the theorem *modulo* the identity (4.5). To get to where we are we have climbed two induction ladders: the n ladder (section 3) and the b ladder (section 4). In order to prove (4.5) we need to climb one more induction ladder: the r-ladder.

5. PROOF OF (4.5): INDUCTION ON r.

In this section n,a,b,c are fixed throughout. As before $t = q^a$, $s = q^b$, $u = q^c$.

Let

$$C_r = C.T.\ [x_1 ... x_r \cdot x^{-\delta} G^{(n)}_{a,b,c}] / H^{(n)}_{a,b,c}, \tag{5.1a}$$

$$D_r = (-q)^r \frac{(t;t)_n (u;t)_r (qs;t)_{n-r}}{(t;t)_r (t;t)_{n-r} (qs;t)_n}, \tag{5.1b}$$

Then (4.5) can be rewritten as

$$C_r = D_r \tag{5.2}$$

Since $C_0 = 1$ by definition and $D_0 = 1$ by plugging r=0 in (5.1b), it follows that (5.2) is true for the base case r=0. The general case would then follow by induction if we can prove that

$$\frac{C_{r+1}}{C_r} = \frac{D_{r+1}}{D_r}. \tag{5.3}$$

A routine calculation shows that

$$\frac{D_{r+1}}{D_r} = -q \cdot \frac{(1-t^{n-r})(1-ut^r)}{(1-t^{r+1})(1-qst^{n-r-1})}. \tag{5.4}$$

Thus we have to prove that

$$\frac{C_{r+1}}{C_r} = -q \cdot \frac{(1-t^{n-r})(1-ut^r)}{(1-t^{r+1})(1-qst^{n-r-1})}. \tag{5.5}$$

It turns out that instead of C_r of (5.1a) it is more convenient to consider

$$A_j = C.T.[x_j ... x_n x^{-\bar{\delta}} G^{(n)}_{a,b,c}], \tag{5.6}$$

where

$$\bar{\delta} = (n-1, ..., 0).$$

But since $\bar{\delta} = rev(\delta)$, where rev is the "reverse permutation" $rev(i) = n - i + 1$, whose sign is $(-1)^{n(n-1)/2)}$,

$$A_j = (-1)^{n(n-1)/2} C.T.[x_1 ... x_{n-j+1} \cdot x^{-\delta} G^{(n)}_{a,b,c}] = (-1)^{n(n-1)/2} C_{n-j+1}. \tag{5.7}$$

It is readily seen that in terms of the A_j (5.5) is equivalent to (take r= n-j+1),

$$\frac{A_{j-1}}{A_j} = -q \frac{(1-t^{j-1})(1-ut^{n-j+1})}{(1-t^{n-j+2})(1-qst^{j-2})}. \tag{5.9}$$

We now go on and prove (5.9).

By using the definitions (2.1),(2.6) and by routine telescoping, we obtain (from now on $G = G^{(n)}_{a,b,c}$, recall that $t = q^a$,$s = q^b$, $u = q^c$).

$$\frac{G(qx_1, ..., x_n)}{G(x_1, ..., x_n)} = \frac{(1-sx_1)\prod_{j=2}^{n}(1-tx_1/x_j)}{(u-x_1)\prod_{j=2}^{n}(q^{-1}t - x_1/x_j)}. \tag{5.10}$$

By cross multiplying we get,

$$(u-x_1)\prod_{j=2}^{n}(q^{-1}t-x_1/x_j)G(qx_1,...,x_n)=(1-sx_1)\prod_{j=2}^{n}(1-tx_1/x_j)G(x_1,...,x_n). \tag{5.11}$$

Expanding the product, we get

$$\sum_{\beta} u(q^{-1}t)^{n-1-|\beta|}(-1)^{|\beta|}x_1^{|\beta|}x^{-\beta}G(qx_1,...,x_n)- \tag{5.12}$$

$$\sum_{\beta}(q^{-1}t)^{n-1-|\beta|}(-1)^{|\beta|}x_1^{|\beta|+1}x^{-\beta}G(qx_1,...,x_n)$$

$$=\sum_{\beta}(-1)^{|\beta|}t^{|\beta|}x_1^{|\beta|}x^{-\beta}G-\sum_{\beta}(-1)^{|\beta|}st^{|\beta|}x_1^{|\beta|+1}x^{-\beta}G,$$

where the sums are over all (0-1) vectors whose first component is zero: $\beta=(0,\beta_2,...\beta_n)$, where $\beta_i=0$ or 1 for $i=2,\ldots,n$.

Let

$$\alpha^{(j)}=(n-1,...,n-j+1,n-j-1,...,-1)=\bar{\delta}-(0_0,...0_{j-1},1_j,...,1_n) \tag{5.13}$$

for $j=2,...,n+1$.

Because of (5.6), we have

$$A^{(j)}=C.T.[x^{-\alpha^{(j)}}G]. \tag{5.14}$$

Multiplying both sides of (5.12) by $x^{-\alpha^{(j)}}$ and taking the constant term, we get (recall that e_1= (1,0,...0))

$$\sum_{\beta} u(q^{-1}t)^{n-1-|\beta|}(-1)^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-|\beta|e_1]}G(qx_1,...,x_n)]- \tag{5.15}$$

$$\sum_{\beta}(q^{-1}t)^{n-1-|\beta|}(-1)^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-(|\beta|+1)e_1]}G(qx_1,...,x_n)$$

$$=\sum_{\beta}(-1)^{|\beta|}t^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-|\beta|e_1]}G]-\sum_{\beta}(-1)^{|\beta|}st^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-(|\beta|+1)e_1]}G].$$

Note that the first component of $\alpha^{(j)}+\beta-|\beta|e_1 is n-1-|\beta|$ and the first component of $\alpha^{(j)}+\beta-(|\beta|+1)e_1$ is $n-2-|\beta|$. Now we use the obvious relation

$$C.T.[x^{-\gamma}G(qx_1,...,x_n)]=q^{\gamma_1}C.T.[x^{-\gamma}G] \tag{5.16}$$

in the left side of (5.15) and we get

$$\sum_{\beta} ut^{n-1-|\beta|}(-1)^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-|\beta|e_1]}G]- \tag{5.17}$$

$$\sum_{\beta} q^{-1}t^{n-1-|\beta|}(-1)^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-(|\beta|+1)e_1]}G]$$

$$=\sum_{\beta}(-1)^{|\beta|}t^{|\beta|}C.T.[x^{-[\alpha^{(j)}+\beta-|\beta|e_1]}G]-\sum_{\beta}(-1)^{|\beta|}t^{|\beta|}sC.T.[x^{-[\alpha^{(j)}+\beta-(|\beta|+1)e_1]}G].$$

We now need the following simple, but crucial, lemma whose proof is left as a pleasant exercise to the reader.

LEMMA

(i) $\alpha^{(j)}+\beta-|\beta|e_1$ is a bad guy unless β has the form $(0,1,...,1,0,...0)$, where the first component is 0 and then for some r, $0 \le r \le j-2$, there are r 1's followed by $n-r-1$ 0's. In this case $\alpha^{(j)}+\beta-|\beta|e_1$ is the image of $\alpha^{(j)}$ under the cycle $(1,2,...,r+1)$, whose sign is $(-1)^r$.

(ii)$\alpha^{(j)}+\beta-(|\beta|+1)e_1$ is a bad guy unless β has the form $(0,1,...1,0,...0)$, where for some r satisfying $j-2 \le r \le n-1$ you have a 0 followed by r 1's followed by n-r-1 0's. In this case $\alpha^{(j)}+\beta-(|\beta|+1)e_1$ is the image of $\alpha^{(j-1)}$ under the cycle (1,2,...,r+1) whose sign is $(-1)^r$.

Discarding all the bad guys in (5.17) and using the above lemma and the *crucial lemma* ,the equation (5.17) shrinks to (recall (5.14))

$$\{\sum_{r=0}^{j-2} ut^{n-1-r}(-1)^r(-1)^r\}A^{(j)}-\{\sum_{r=j-2}^{n-1} q^{-1}t^{n-1-r}(-1)^r(-1)^r\}A_{(j-1)}$$

$$=\{\sum_{r=0}^{j-2}(-1)^r t^r(-1)^r\}A^{(j)}-\{\sum_{r=j-2}^{n-1}(-1)^r t^r s(-1)^r\}A_{(j-1)} \tag{5.18}$$

By summing all the geometric series and performing very routine and simple ninth grade algebra we get (5.9). tav vav shin lamed bet ayin

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