

52 Strange Evaluations by Shalosh B. Ekhad

Theorem 1 :

$$F(-n, -4n - 1/2, -3n, -1) = (64/27)^n \cdot \frac{(3/8)_n (5/8)_n}{(1/3)_n (2/3)_n} .$$

Proof :

$$\frac{-2(296n^3 - 216n^2k + 771n^2 + 651n - 372nk + 52nk^2 - 155k + 177 - 4k^3 + 45k^2)(3n - k + 1)k}{(8n + 9 - 2k)(8n + 7 - 2k)(8n + 5 - 2k)(8n + 3 - 2k)(n - k + 1)} \quad \square$$

Theorem 2 :

$$F(-n, -4n - 5/2, -3n - 1, -1) = (64/27)^n \cdot \frac{(7/8)_n (9/8)_n}{(2/3)_n (4/3)_n} .$$

Proof :

$$\frac{(296n^3 + 1215n^2 - 216n^2k + 1660n + 52nk^2 - 586nk + 756 - 394k - 4k^3 + 71k^2) \cdot (-2(3n - k + 2)k)}{(8n + 13 - 2k)(8n + 11 - 2k)(8n + 9 - 2k)(8n + 7 - 2k)(n - k + 1)} \quad \square$$

Theorem 3 :

$$F(-n, -3n - 1/2, -4n, 4) = (-27/16)^n \cdot \frac{(1/3)_n (5/6)_n}{(1/4)_n / (3/4)_n} .$$

Proof :

$$\frac{(344n^3 - 216n^2k + 768n^2 + 522n - 282nk + 34nk^2 - 75k + 104 + 13k^2)(4n - k + 1)k}{4(6n + 7 - 2k)(6n + 5 - 2k)(6n + 3 - 2k)(n - k + 1)(3n + 1)} \quad \square$$

Theorem 4 :

$$F(-n, -3n - 3/4, -4n - 1, -8) = (27/4)^n \cdot \frac{(2/3)_n (7/12)_n}{(1/2)_n (3/4)_n} .$$

Proof :

$$\frac{-(2208n^3 - 1488n^2k + 6272n^2 + 5762n - 2620nk + 252nk^2 - 1082k + 1707 + 167k^2)(4n - k + 2)k}{12(12n + 15 - 4k)(12n + 11 - 4k)(12n + 7 - 4k)(n - k + 1/(3n + 2))} \quad \square$$

Theorem 5 :

$$F(-n, -3n - 3/2, -4n - 2, -8) = (27/4)^n \cdot \frac{(5/6)_n (7/6)_n}{(3/4)_n (5/4)_n} .$$

Proof :

$$-(1/24) \frac{(184n^2 + 492n - 124nk + 326 + 21k^2 - 167k)(4n - k + 3)k}{(6n + 9 - 2k)(6n + 7 - 2k)(6n + 5 - 2k)(n - k + 1)} \quad \square$$

Theorem 6 :

$$F(-n, -3n - 1/4, -4n, -8) = (27/4)^n \cdot \frac{(1/3)_n (5/12)_n}{(1/2)_n (1/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3 - 496n^2k + 1600n^2 + 1058n + 84nk^2 - 624nk - 149k + 200 + 27k^2)(4n - k + 1)k}{(12n + 13 - 4k)(12n + 9 - 4k)(12n + 5 - 4k)(n - k + 1)(3n + 1)} \quad \square$$

Theorem 7 :

$$F(-n, -3n - 1/2, -4n, -8) = (27/4)^n \cdot \frac{(1/3)_n (2/3)_n}{(1/4)_n (3/4)_n}.$$

Proof :

$$\frac{(-1/32)(2208n^4 - 1488n^3k + 6640n^3 + 252n^2k^2 - 2988n^2k + 7104n^2 + 248nk^2 - 1812nk + 3156n + 53k^2 - 321k + 478) \cdot (4n - k + 1)k}{(6n + 7 - 2k)(6n + 5 - 2k)(6n + 3 - 2k)(n - k + 1)(3n + 2)(3n + 1)} \quad \square$$

Theorem 8 :

$$F(-n, -3n - 5/4, -4n - 1, -8) = (27/4)^n \cdot \frac{(2/3)_n (13/12)_n}{(1/2)_n (5/4)_n}.$$

Proof :

$$\frac{(-1/4)(736n^3 - 496n^2k + 2336n^2 + 2382n - 956nk + 84nk^2 - 414k + 767 + 55k^2)(4n - k + 2)k}{(12n + 17 - 4k)(12n + 13 - 4k)(12n + 9 - 4k)(n - k + 1)(3n + 2)} \quad \square$$

Theorem 9 :

$$F(-n, -3n - 3/2, -4n - 2, 4) = (-27/16)^n \cdot \frac{(4/3)_n (5/6)_n}{(3/4)_n (5/4)_n}.$$

Proof :

$$\frac{(1/12)(1032n^3 - 648n^2k + 4024n^2 + 5202n - 1690nk + 102nk^2 - 1103k + 2228 + 135k^2)(4n - k + 3)k}{(6n + 9 - 2k)(6n + 7 - 2k)(6n + 5 - 2k)(n - k + 1)(3n + 4)} \quad \square$$

Theorem 10 :

$$F(-n, -3n - 7/4, -4n - 2, -8) = (27/4)^n \cdot \frac{(4/3)_n (11/12)_n}{(3/2)_n (3/4)_n}.$$

Proof :

$$\frac{(-1/12)(2208n^3 + 9216n^2 - 1488n^2k + 12806n + 252nk^2 - 4112nk + 5924 - 2839k + 335k^2)(4n - k + 3)k}{(12n + 19 - 4k)(12n + 15 - 4k)(12n + 11 - 4k)(n - k + 1)(3n + 4)} \quad \square$$

Theorem 11 : Let ω be a primitive cubic root of unity (i.e. $\omega = (1 \pm \sqrt{3}i)/2$), then

$$F(-n, -3n - 1, -2n, \omega) = (-(3/2)\omega + 3/4)^n \cdot \frac{(2/3)_n}{(1/2)_n}.$$

Proof :

$$(129n^2 + 289n - 2\omega n + 160 - 2\omega - 88kn + 14k\omega n - 101k + 19k\omega + 13k^2 - 5\omega k^2) \cdot$$

$$\frac{(-1/129)(-5 + 13\omega)(2n - k + 1)k}{(3n + 4 - k)(3n + 3 - k)(3n + 2 - k)(n - k + 1)} \quad \square$$

Theorem 12 :

$$F(-n, -3n - 1/2, -n + 1/2, 4) = (-27)^n.$$

Proof :

$$\frac{(1/9)k(2n + 1 - 2k)(84n^2 + 172n - 60nk + 12k^2 - 64k + 87)}{(6n + 7 - 2k)(6n + 5 - 2k)(6n + 3 - 2k)(n + 1 - k)} \quad \square$$

Theorem 13 :

$$F(-n, -3n - 3/2, -n + 1/2, 4) = (-27)^n \cdot \frac{(5/6)_n(7/6)_n}{(1/2)_n(3/2)_n}.$$

Proof :

$$\frac{(1/3)(28n^2 + 76n - 20nk + 51 - 28k + 4k^2)(2n + 1 - 2k)k}{(6n + 9 - 2k)(6n + 7 - 2k)(6n + 5 - 2k)(n + 1 - k)} \quad \square$$

Theorem 14 : Let $\alpha = 9 \pm 4\sqrt{5}$, then

$$F(-n, -3n - 1/2, 2n + 3/2, \alpha) = (128/125 + 128/125\alpha)^n \cdot \frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n}.$$

Proof :

$$(79794 + 388827n + 706989n^2 + 566136n^3 + 167796n^4 - 4518k - 4875nk - 51972k^2 + 11429\alpha nk + 3754\alpha k + 2132\alpha k^2 + 496k^4\alpha + 26472k^3 - 3696k^4 - 6816kn^3 - 6174kn^2 - 127434k^2n - 72852k^2n^2 + 30768k^3n - 3336n^3\alpha - 9466\alpha n^2 - 8839\alpha n - 2718\alpha + 10998k\alpha n^2 + 1060k^2\alpha n^2 - 2192k^3\alpha n + 3336kn^3\alpha + 3058k^2\alpha n - 2584k^3\alpha).$$

$$\frac{-(1/74576)(31\alpha - 327)k}{(6n + 7 - 2k)(6n + 5 - 2k)(6n + 3 - 2k)(n - k + 1)(4n + 3 + 2k)(\alpha - 1)}. \quad \square$$

Theorem 15 : Let $\alpha = 9 \pm 4\sqrt{5}$, then

$$F(-n, -3n - 3/2, 2n + 5/2, \alpha) = (128/125 + 128/125\alpha)^n \cdot \frac{(5/4)_n (7/4)_n}{(7/5)_n (8/5)_n}.$$

Proof :

$$\begin{aligned} & (384115 + 1282296n + 1575413n^2 + 845796n^3 + 167796n^4 - 86193k - 153105nk - 77610k^2 + 26963\alpha nk + \\ & \quad 13383\alpha k + 730\alpha k^2 + 496k^4\alpha + 34264k^3 - 3696k^4 - 6816kn^3 - \\ & \quad 74602kn^2 - 154458k^2n - 72852k^2n^2 + 30768k^3n - 3336n^3\alpha - 12126\alpha n^2 - 14517\alpha n - \\ & \quad 5745\alpha + 17114k\alpha n^2 + 1060k^2\alpha n^2 - 2192k^3\alpha n + 3336kn^3\alpha + 1842k^2\alpha n - 2984k^3\alpha) \cdot \\ & \quad \frac{k(-1/74576)(31\alpha - 327)}{(6n + 9 - 2k)(6n + 7 - 2k)(6n + 5 - 2k)(n - k + 1)(4n + 5 + 2k)(\alpha - 1)}. \quad \square \end{aligned}$$

Theorem 16 : Let $\alpha = 2 \pm \sqrt{3}$, then

$$F(-n, -2n - 1/2, -4n - 1, 4\alpha) = (3/4 - 3/2\alpha)^n \cdot \frac{(2/3)_n}{(1/2)_n}.$$

Proof :

$$\begin{aligned} & (488n^2 + 978n - 20\alpha n + 472 - 23\alpha + 94k^2 - 25k^2\alpha - 420kn + 52kn\alpha - 422k + 72k\alpha) \cdot \\ & \quad \frac{(1/732)(-68 + 19\alpha)(4n + 2 - k)k}{(4n + 5 - 2k)(4n + 3 - 2k)(n - k + 1)(3n + 2)}. \quad \square \end{aligned}$$

Theorem 17 :

$$F(-n, -2n - 1/3, -3n, 9) = (-4)^n \cdot \frac{(1/2)_n}{(1/3)_n}.$$

Proof :

$$\frac{(1/9)k(3n - k + 1)(45n^2 - 21nk + 72n - 11k + 25)}{(6n + 7 - 3k)(6n + 4 - 3k)(n - k + 1)(2n + 1)}. \quad \square$$

Theorem 18 :

$$F(-n, -2n - 2/3, -3n - 1, 9) = (-4)^n \cdot \frac{(5/6)_n}{(2/3)_n}.$$

Proof :

$$\frac{(1/6)(15n + 19 - 7k)(3n - k + 2)k}{(6n + 8 - 3k)(6n + 5 - 3k)(n - k + 1)}. \quad \square$$

Theorem 19 :

$$F(-n, -n - 1/2, 4n + 9/2, 1/5) = \left(\frac{16384}{15625}\right)^n \cdot \frac{(9/8)_n(11/8)_n(13/8)_n(15/8)_n}{(6/5)_n(9/5)_n(13/10)_n(17/10)_n} .$$

Proof :

$$\begin{aligned} & (3036n^4 + 1256n^3k + 16960n^3 + 35275n^2 + 4780n^2k - 1184n^2k^2 + 6180nk - \\ & 3440nk^2 - 544nk^3 + 32445n + 2720k + 11154 - 2480k^2 - 64k^4 - 800k^3) \cdot \\ & \frac{(-5/16)k}{(2n + 3 - 2k)(n - k + 1)(8n + 13 + 2k)(8n + 11 + 2k)(8n + 9 + 2k)} \quad \square \end{aligned}$$

Theorem 20 :

$$F(-n, -n + 1/2, 4n + 3/2, 1/5) = \left(\frac{16384}{15625}\right)^n \cdot \frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(3/5)_n(2/5)_n(9/10)_n(11/10)_n} .$$

Proof :

$$\begin{aligned} & (3036n^4 + 1256n^3k + 7852n^3 + 7409n^2 + 2844n^2k - 1184n^2k^2 + 2252nk - 1344nk^2 + 2993n - \\ & 544nk^3 - 336k^2 + 592k + 430 - 352k^3 - 64k^4) \cdot \\ & \frac{(-5/16)k}{(2n + 1 - 2k)(n - k + 1)(8n + 7 + 2k)(8n + 5 + 2k)(8n + 3 + 2k)} \quad \square \end{aligned}$$

Theorem 21 : Let $\beta = (3 \pm \sqrt{5})/2$, then

$$F(-n, 2n + 1, -3n, \beta) = ((25/27) + (25/27)\beta)^n \cdot \frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n} .$$

Proof :

$$\begin{aligned} & (1376n + 308 + 2026n^2 + 958n^3 - 60k - 23k^3\beta + 38k^2\beta - 91k\beta + 139\beta n + 87\beta n^2 \\ & - 201k\beta n - 87k\beta n^2 + 38k^2\beta n + 48k^3 + 4k^2 - 122kn - 110kn^2 + 4k^2n + 52\beta) \cdot \\ & \frac{(1/4790)k(3n - k + 1)(-78 + 17\beta)}{(n - k + 1)(5n + 3)(5n + 2)(n + 1)(2n + 1)} \quad \square \end{aligned}$$

Theorem 22 : Let $\beta = (3 \pm \sqrt{5})/2$, then

$$F(-n, 2n + 2, -3n - 1, \beta) = (25/27 + 25/27\beta)^n \cdot \frac{(4/5)_n(6/5)_n}{(2/3)_n(4/3)_n} .$$

Proof :

$$(4179n + 3463n^2 + 958n^3 + 24k - 317k\beta n - 87k\beta n^2 + 48k^3 + 54k^2 - 109kn - 110kn^2 + 4k^2n + 96\beta + 1674 + 183\beta n + 87\beta n^2 + 38k^2\beta n - 23k^3\beta + 34k^2\beta - 251k\beta) \cdot \frac{(1/4790)k(3n - k + 2)(-78 + 17\beta)}{(n - k + 1)(5n + 6)(5n + 4)(2n + 3)(n + 1)} \quad \square$$

Theorem 23 : Let ϕ be the Golden ratio $(1 + \sqrt{5})/2$ then

$$F(-n, 2n + 1, 4n + 2, \phi^{-1}) = (128/125 - 64/125\phi^{-1})^n \cdot \frac{(3/4)_n(5/4)_n}{(4/5)_n(6/5)_n}.$$

Proof :

$$(2815n + 3661n^2 + 1558n^3 + 14k^3\phi^{-1} - 148\phi^{-1}n - 84\phi^{-1}n^2 + 579k + 183k\phi^{-1}n + 1567kn + 1013kn^2 + 88k\phi^{-1} + 37k^2\phi^{-1}n + 34k^2\phi^{-1} + 270k^2n + 84k\phi^{-1}n^2 + 712 + 206k^2 + 39k^3 - 64\phi^{-1}) \cdot \frac{(-1/779)k(14\phi^{-1} - 25)}{(n - k + 1)(4n + 4 + k)(4n + 3 + k)(4n + 2 + k)(-1 + \phi^{-1})} \quad \square$$

Theorem 24 : Let ϕ be the Golden ratio $(1 + \sqrt{5})/2$ then

$$F(-n, 2n + 2, 4n + 4, \phi^{-1}) = (128/125 - 64/125\phi^{-1})^n \cdot \frac{(5/4)_n(7/4)_n}{(7/5)_n(8/5)_n}.$$

Proof :

$$(7628n + 5998n^2 + 1558n^3 + 14k^3\phi^{-1} - 178\phi^{-1}n - 84\phi^{-1}n^2 + 3188 + 1762k + 309k\phi^{-1}n + 2697kn + 1013kn^2 + 253k\phi^{-1} + 37k^2\phi^{-1}n + 79k^2\phi^{-1} + 270k^2n + 84k\phi^{-1}n^2 + 387k^2 + 39k^3 - 94\phi^{-1}) \cdot \frac{(-1/779)k(14\phi^{-1} - 25)}{(n - k + 1)(4n + 6 + k)(4n + 5 + k)(4n + 4 + k)(-1 + \phi^{-1})} \quad \square$$

Theorem 25 : Let $\gamma = 3 \pm 2\sqrt{2}$, then

$$F(-2n, -4n - 1/2, -2n + 1/2, \gamma) = (-16 + 96\gamma)^n \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n}.$$

Proof :

$$(25434 + 137162n + 271600n^2 + 233440n^3 + 73472n^4 + 5216k^2\gamma n^2 + 10144k^2\gamma n - 21148k\gamma n -$$

$$\begin{aligned}
& 6208k\gamma n^3 - 20304k\gamma n^2 - 1264k^3\gamma n - 142224kn^2 - 58944kn^3 - 111324kn + 14304k^2n^2 + \\
& 22536k^2n - 1776k^3n - 1372k^3 + 8594k^2 + 88k^4 + 4834k^2\gamma - 7067k\gamma - 1204k^3\gamma - 28145k + 3837\gamma + \\
& 104k^4\gamma + 13852\gamma n + 6208\gamma n^3 + 16240\gamma n^2). \\
& \frac{(-1/1148)(-89 + 13\gamma)(4n + 1 - 2k)k}{(8n + 9 - 2k)(8n + 7 - 2k)(8n + 5 - 2k)(8n + 3 - 2k)(2n - k + 1)(2n - k + 2)(\gamma - 1)} \quad \square
\end{aligned}$$

Theorem 26 :

$$F(-2n, -4n - 1/3, 2/3, -8) = 729^n.$$

Proof :

$$\begin{aligned}
& (314496n^4 + 1101600n^3 + 1416600n^2 + 789644n + 160228 - 1012068kn^2 - 393984kn^3 - \\
& 846054kn - 229356k + 172044k^2n^2 + 290142k^2n + 119151k^2 - 32076k^3n - 26730k^3 + 2187k^4). \\
& \frac{(1/729)(-1 + 3k)k}{(2n - k + 1)(2n - k + 2)(12n + 4 - 3k)(12n + 7 - 3k)(12n + 10 - 3k)(12n + 13 - 3k)} \quad \square
\end{aligned}$$

Theorem 27 : Let $\delta = 5 \pm 3\sqrt{3}$, then

$$F(-2n, -4n - 1/3, 2/3, 2\delta) = (702\delta + 135)^n \cdot \frac{(7/12)_n}{(3/4)_n}.$$

Proof :

$$\begin{aligned}
& (2059512 + 12099448n + 766292\delta n + 25671232n^2 + 23410464n^3 + 7767936n^4 + 933000\delta n^2 - \\
& 1239770k\delta n - 404352kn^3\delta + 506562k^2\delta n + 268020k^2\delta n^2 - 62100k^3\delta n - 1257180k\delta n^2 \\
& - 2121648k - 5589216kn^3 - 12542136kn^2 - 9090980kn + 1684440k^2n^2 + 2508684k^2n \\
& - 240624k^3n - 56934k^3\delta + 232953k^2\delta - 390660k\delta + 370656n^3\delta \\
& + 205020\delta + 13446k^4 - 178812k^3 + 904542k^2 + 4941k^4\delta). \\
& \frac{(1/364122)k(-1 + 3k)(61\delta - 776)}{(12n + 13 - 3k)(12n + 10 - 3k)(12n + 7 - 3k)(12n + 4 - 3k)(2n - k + 1)(2n - k + 2)2\delta - 1)} \quad \square
\end{aligned}$$

Theorem 28 : Let $\delta = 5 \pm 3\sqrt{3}$, then

$$F(-2n, -4n - 2/3, 4/3, 2\delta) = (702\delta + 135)^n \cdot \frac{(5/12)_n}{(5/4)_n}.$$

Proof :

$$\begin{aligned}
& (3710744 + 18504312n + 607916\delta n + 33746032n^2 + 26726016n^3 + 7767936n^4 + 661656\delta n^2 - \\
& 1315958k\delta n - 404352kn^3\delta + 533802k^2\delta n + 268020k^2\delta n^2 - 62100k^3\delta n - 1280676k\delta n^2 -
\end{aligned}$$

$$\begin{aligned}
& 3169096k - 5589216kn^3 - 14213784kn^2 - 11770892kn + 1684440k^2n^2 + 2829540k^2n - 240624k^3n \\
& - 60696k^3\delta + 260835k^2\delta - 439228k\delta + 235872n^3\delta + 182668\delta \\
& + 13446k^4 - 200988k^3 + 1159494k^2 + 4941k^4\delta) \cdot \\
& \frac{(1/364122)k(1+3k)(61\delta-776)}{(12n+14-3k)(12n+11-3k)(12n+8-3k)(12n+5-3k)(2n-k+1)(2n-k+2)(2\delta-1)} \quad \square
\end{aligned}$$

Theorem 29 :

$$F(-2n, -4n - 2/3, 4/3, -8) = 729^n \cdot \frac{(5/12)_n(11/12)_n}{(3/4)_n(5/4)_n}.$$

Proof :

$$\begin{aligned}
& (34944n^4 - 43776kn^3 + 131104n^3 - 121596kn^2 + 181768n^2 + 19116k^2n^2 - \\
& 110586kn + 110164n + 35046k^2n - 3564k^3n + 15741k^2 + 243k^4 - 32868k + 24564 - 3240k^3) \cdot \\
& \frac{(1/81)(1+3k)k}{(12n+14-3k)(12n+11-3k)(12n+8-3k)(12n+5-3k)(2n-k+1)(2n-k+2)} \quad \square
\end{aligned}$$

Theorem 30 : Let $\alpha = 2 \pm \sqrt{3}$, then

$$F(-2n, -3n - 1/4, -4n, 4\alpha) = (39/2\alpha - 21/4)^n \cdot \frac{(5/12)_n}{(1/4)_n}.$$

Proof :

$$\begin{aligned}
& (171978n + 238000n^2 + 104736n^3 - 12496k\alpha n^2 - 32440k\alpha n + 7780k^2\alpha n + 17488\alpha n^2 + \\
& 30900\alpha n + 8407k^2\alpha - 1216k^3\alpha - 18581k\alpha + 39182 + 13262\alpha \\
& - 81808kn^2 + 15490k^2 - 1792k^3 - 43403k - 122224kn + 21112k^2n) \cdot \\
& \frac{(-1/3273)k(4n-k+1)(397\alpha-1484)}{(12n+13-4k)(12n+9-4k)(12n+5-4k)(2n-k+1)(2n-k+2)} \quad \square
\end{aligned}$$

Theorem 31 :

$$F(-2n, -3n + 1, -2n + 2, -2) = 27^n \cdot \frac{(1/2)_n}{(-1/2)_n}.$$

Proof :

$$\frac{(-1/9)k(2n-k-1)(39n^2+55n-21kn-14k+3k^2+18)}{(3n+2-k)(3n+1-k)(3n-k)(2n+2-k)} \quad \square$$

Theorem 32 :

$$F(-2n, -3n, 2n + 1/2, 1/4) = (27/64)^n \cdot \frac{(1/3)_n(2/3)_n}{(1/4)_n(3/4)_n}.$$

Proof :

$$\frac{(-2/3)(74n^3 + 168n^2 - 57kn^2 - 87kn + 124n + 12k^2n - 32k + 30 - k^3 + 9k^2)(4n + 1 - 2k)k}{(3n + 3 - k)(3n + 2 - k)(3n + 1 - k)(2n + 1 - k)(2n + 2 - k)} \quad \square$$

Theorem 33 : Let $\eta = -7 \pm 4\sqrt{3}$, then

$$F(-2n, -3n - 1/4, -n + 3/4, \eta) = (312\eta + 24)^n \cdot \frac{(5/12)_n}{(1/4)_n} .$$

Proof :

$$\begin{aligned} & (485694n + 611008n^2 + 248736n^3 - 5600k^2\eta n + 11504k\eta n^2 + 23720\eta nk - 11880\eta n - \\ & 6512\eta n^2 + 752k^3\eta + 22744k^2 + 11851\eta k - 5432\eta k^2 - 204544nk - 79751k - 2416k^3 \\ & + 28576k^2n - 126928kn^2 - 5299\eta + 124358) \cdot \\ & \frac{(1/124368)(47\eta + 809)(4n + 1 - 4k)k}{(12n + 13 - 4k)(12n + 9 - 4k)(12n + 5 - 4k)(2n - k + 1)(2n - k + 2)(\eta - 1)} \quad \square \end{aligned}$$

Theorem 34 :

$$F(-2n, -2n + 1/3, -n + 5/6, -1/8) = (27/16)^n \cdot \frac{(1/3)_n}{(1/6)_n} .$$

Proof :

$$\frac{(-4/27)(132n^2 - 60nk + 202n + 74 + 9k^2 - 45k)(6n + 1 - 6k)k}{(6n + 5 - 3k)(6n + 2 - 3k)(2n - k + 1)(2n - k + 2)} \quad \square$$

Theorem 35 :

$$F(-2n, -2n + 1/2, 4n + 3/2, -1/3) = (4096/6561)^n \cdot \frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(1/3)_n/(5/6)_n/(7/12)_n/(13/12)_n} .$$

Proof :

$$\begin{aligned} & (21483 + 241989n + 1106816n^2 + 2625708n^3 + 3409136n^4 + 2299968n^5 + 631040n^6 - \\ & 405096kn^2 - 723616kn^3 - 213504kn^5 - 628608kn^4 - 161808k^2n^2 - 175872k^2n^3 - 72960k^2n^4 - 10936k^2 \\ & - 109724kn + 31744k^3n^3 + 5856k^3 - 67368k^2n - 768k^5 + 5376k^4n^2 + 6720k^4n + 2624k^4 \\ & + 31136k^3n + 54144k^3n^2 - 256k^6 - 1536k^5n - 11388k) \cdot \\ & \frac{(-3/32)k}{(4n + 3 - 2k)(4n + 1 - 2k)(2n - k + 2)(2n - k + 1)(8n + 7 + 2k)(8n + 5 + 2k)(8n + 3 + 2k)} \quad \square \end{aligned}$$

Theorem 36 :

$$F(-2n, -2n + 1/4, 4n + 5/4, 1/9) = (65536/59049)^n \cdot \frac{(9/16)_n (5/16)_n (13/16)_n (17/16)_n}{(1/3)_n (5/6)_n (13/24)_n (25/24)_n}.$$

Proof :

$$\begin{aligned} & (501813 + 5630279n + 25382756n^2 + 59064356n^3 + 75057056n^4 + 49523456n^5 + 13285376n^6 + 1953360kn^2 \\ & - 59200kn^3 - 1363968kn^5 - 2306048kn^4 - 9668544k^2n^2 - 9942016k^2n^3 - 3846144k^2n^4 \\ & - 684064k^2 + 1280440kn + 753664k^3n^3 + 36608k^3 - 4189920k^2n \\ & - 16384k^5 + 368640k^4n^2 + 523264k^4n + 195584k^4 + 446720k^3n + 1058816k^3n^2 - 16384k^6 - 49152k^5n + 247832k) \cdot \\ & \frac{(-9/128)k}{(8n + 7 - 4k)(8n + 3 - 4k)(2n - k + 2)(2n - k + 1)(16n + 13 + 4k)(16n + 9 + 4k)(16n + 5 + 4k)} \quad \square \end{aligned}$$

Theorem 37 :

$$F(-2n, -2n + 1/2, 4n + 3/2, 1/9) = (65536/59049)^n \cdot \frac{(3/8)_n (7/8)_n (5/8)_n (9/8)_n}{(7/12)_n (5/12)_n (11/12)_n (13/12)_n}.$$

Proof :

$$\begin{aligned} & (693315 + 7099938n + 29906648n^2 + 66047424n^3 + 80454656n^4 + 51184128n^5 + 13285376n^6 \\ & - 3206400kn^2 - 5387776kn^3 - 1363968kn^5 - 4392960kn^4 - 9505536k^2n^2 - \\ & 9778176k^2n^3 - 3846144k^2n^4 - 696736k^2 - 960464kn + 753664k^3n^3 + 218880k^3 \\ & - 4165824k^2n - 36864k^5 + 368640k^4n^2 + 491520k^4n + 181760k^4 + 935936k^3n + 1413120k^3n^2 \\ & - 16384k^6 - 49152k^5n - 121896k) \cdot \\ & \frac{(-9/4096)k}{(4n + 3 - 2k)(4n + 1 - 2k)(2n - k + 2)(2n - k + 1)(8n + 7 + 2k)(8n + 5 + 2k)(8n + 3 + 2k)} \quad \square \end{aligned}$$

Theorem 38 :

$$F(-2n, -2n - 1/4, 4n + 11/4, 1/9) = (65536/59049)^n \cdot \frac{(11/16)_n (15/16)_n (19/16)_n (23/16)_n}{(4/3)_n (5/6)_n (19/24)_n (31/24)_n}.$$

Proof :

$$\begin{aligned} & (6391980 + 44661417n + 128402584n^2 + 194585468n^3 + 163941024n^4 + 72772864n^5 + 13285376n^6 - 18793632kn^2 \\ & - 17944128kn^3 - 1363968kn^5 - 8184832kn^4 - 17754048k^2n^2 - 13460480k^2n^3 \\ & - 3846144k^2n^4 - 2275184k^2 - 9616576kn + 753664k^3n^3 + 846464k^3 - 10406208k^2n - 69632k^5) \end{aligned}$$

$$\frac{+368640k^4n^2+644096k^4n+281088k^4+2419968k^3n+2332672k^3n^2-16384k^6-49152k^5n-1949232k}{(-9/128)k} \cdot \square$$

Theorem 39 : Let $\zeta = -1 \pm \sqrt{2}$, then

$$F(-2n, 1/2, -4n, 2\zeta) = \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n} .$$

Proof :

$$\frac{(-1/2)(3+\zeta)(6\zeta n-2n+4\zeta-3k^2-k^2\zeta+8kn+6k)(4n-k+1)k}{(2n-k+2)(2n-k+1)(8n+5)(8n+3)} \square$$

Theorem 40 :

$$F(-2n, 1/2, 4n+5/2, 1/4) = \frac{(7/8)_n(5/8)_n(9/8)_n(11/8)_n}{(3/4)_n^2(5/4)_n^2} .$$

Proof :

$$\frac{(-4/3)(1+2k)(256n^3+640n^2-24k^2n+550n-4k^3-24k^2+k+162)k}{(2n-k+2)(2n-k+1)(8n+9+2k)(8n+7+2k)(8n+5+2k)} \square$$

Theorem 41 :

$$F(-2n, 1/2, 4n+3/2, 1/4) = \frac{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}{(7/12)_n(5/12)_n(11/12)_n(13/12)_n} .$$

Proof :

$$\frac{(-4/81)(-1+6k)(2304n^3+4608n^2-216k^2n+144kn+3214n-36k^3-144k^2+117k+770)k}{(2n-k+2)(2n-k+1)(8n+7+2k)(8n+5+2k)(8n+3+2k)} \square$$

Theorem 42 : Let ω be a primitive cubic root of unity (i.e. $\omega = (1 \pm \sqrt{3}i)/2$), then

$$F(-2n, 2n+1, -4n, \omega) = ((27/16)\omega)^n \cdot \frac{(1/3)_n(5/6)_n}{(1/4)_n(3/4)_n} .$$

Proof :

$$\begin{aligned} &(2660+14784n+30156n^2+26880n^3+8848n^4+308\omega+1316\omega n-11k^4\omega-16k^4-1936kn^3\omega-4852kn^2\omega-4086kn\omega \\ &+204k^3+1234k^2\omega n+708k^2\omega n^2-526k^2-6052kn^2-2816kn^3+236k^3n-4234kn+2k^3\omega-1148\omega k+537\omega k^2-780k^2n^2 \\ &+24k^3\omega n-1322k^2n+1792\omega n^2+784\omega n^3-966k) \cdot \end{aligned}$$

$$\frac{(1/6636)k(4n - k + 1)(-27 + 11\omega)}{(2n + 1)(2n - k + 2)(2n - k + 1)(6n + 5)(3n + 1)(n + 1)} \quad \square$$

Theorem 43 : Let ϕ be the Golden ratio $(1 + \sqrt{5})/2$ then

$$F(-2n, 2n + 1, -3n + 1/2, \phi) = ((100/27)\phi + 25/9)^n \cdot \frac{(3/10)_n(7/10)_n}{(1/6)_n(5/6)_n}.$$

Proof :

$$\begin{aligned} & (1678n + 2708n^2 + 1448n^3 + 136k^2\phi n - 264k\phi n^2 - 364k\phi n - 118k\phi + 324\phi n \\ & + 264\phi n^2 - 26k^3\phi + 96k^2\phi + 343 + 96\phi - 267k - 19k^2 + 24k^3 - 42k^2n - 536kn^2 - 750kn) \cdot \\ & \frac{(2/905)k(6n + 1 - 2k)(-43 + 16\phi)}{(2n + 1)(2n - k + 2)(2n - k + 1)(10n + 7)(10n + 3)} \quad \square \end{aligned}$$

Theorem 44 : Let ϕ be the Golden ratio $(1 + \sqrt{5})/2$ then

$$F(-2n, 2n + 1, -3n, \phi) = (100/27\phi + 25/9)^n \cdot \frac{(2/5)_n(3/5)_n}{(1/3)_n(2/3)_n}.$$

Proof :

$$\begin{aligned} & (1334n + 1716n^2 + 724n^3 - 270k\phi n + 68k^2\phi n - 132k\phi n^2 - 125k\phi + 214\phi n + 132\phi n^2 \\ & - 13k^3\phi + 68k^2\phi + 82\phi - 177k - 21k^2 + 12k^3 - 21k^2n - 268kn^2 - 433kn + 342) \cdot \\ & \frac{(1/905)k(3n - k + 1)(-43 + 16\phi)}{(2n - k + 2)(2n - k + 1)(5n + 3)(5n + 2)(n + 1)} \quad \square \end{aligned}$$

Theorem 45 : Let $\zeta_1 = 1 \pm \sqrt{2}$, then

$$F(-2n, 2n + 1, -2n + 1/2, (1/2)\zeta_1) = (4 + 8\zeta_1)^n \cdot \frac{(3/8)_n(5/8)_n}{(1/4)_n(3/4)_n}.$$

Proof :

$$\begin{aligned} & (56n^2 + 72n + 8\zeta_1n + 24 + 5\zeta_1 - 6k^2 - 16kn - 12k + 4k^2\zeta_1 - 8\zeta_1kn - 6\zeta_1k) \cdot \\ & \frac{(2/7)k(4n + 1 - 2k)(-4 + \zeta_1)}{(2n - k + 2)(2n - k + 1)(8n + 5)(8n + 3)} \quad \square \end{aligned}$$

Theorem 46 : Let $\alpha = 2 \pm \sqrt{3}$, then

$$F(-2n, 2n + 1, -n + 3/4, (1/4)\alpha) = (3 - 6\alpha)^n \cdot \frac{(5/12)_n}{(1/4)_n}.$$

Proof :

$$\frac{(1/3)(8n + 5 + \alpha - 2k\alpha + 2k)(-5 + \alpha)(4n + 1 - 4k)k}{(2n - k + 2)(2n - k + 1)(12n + 5)} \quad \square$$

Theorem 47 :

$$F(-2n, 3n + 1, -2n + 2, 2/5) = (125/27)^n \cdot \frac{(1/2)_n}{(-1/2)_n}.$$

Proof :

$$\frac{(-1/25)(147n^2 + 33nk + 179n + 3k^2 + 23k + 50)(2n - k - 1)k}{n(2n + 2 - k)(3n + 2)(3n + 1)} \quad \square$$

Theorem 48 :

$$F(-2n, 4n + 1, -4n + 1/2, 2) = 16^n \cdot \frac{(5/16)_n(3/16)_n(13/16)_n(11/16)_n}{(3/8)_n(1/8)_n(7/8)_n(5/8)_n}.$$

Proof :

$$\begin{aligned} & (20886 + 229992n + 1021408n^2 + 2335744n^3 + 2908160n^4 \\ & + 1875968n^5 + 491520n^6 - 852992kn^4 - 1028864kn^3 - 278528kn^5 \\ & - 611776kn^2 + 86016k^2n^3 + 30720k^2n^4 + 43152k^2n - 5344k^3n - 5760k^3n^2 \\ & - 2048k^3n^3 - 1583k^3 + 91168k^2n^2 \\ & + 278k^4 + 480k^4n + 128k^4n^2 - 68k^5 + 8k^6 - 64k^5n + 7628k^2 - 179848nk - 21029k) \cdot \\ & \frac{(-1/4)(8n + 1 - 2k)k}{(2n - k + 2)(2n - k + 1)(16n + 11)(16n + 13)(16n + 3)(16n + 5)(4n + 3)(4n + 1)} \quad \square \end{aligned}$$

Theorem 49 :

$$F(-2n, 4n + 2, -4n - 1/2, 2) = 16^n \cdot \frac{(9/16)_n(7/16)_n(15/16)_n(17/16)_n}{(3/8)_n(7/8)_n(5/8)_n(9/8)_n}.$$

Proof :

$$\begin{aligned} & (268890 + 1832424n + 5143712n^2 + 7618816n^3 + 6283264n^4 + 2736128n^5 + 491520n^6 - 1309696kn^4 \\ & - 2438912kn^3 - 278528kn^5 - 2249600kn^2 + 125952k^2n^3 + 30720k^2n^4 + 134312k^2n - 10400k^3n - 8064k^3n^2 - 2048k^3n^3 - \\ & 4313k^3 + 194848k^2n^2 + 518k^4 + 640k^4n + 128k^4n^2 - 92k^5 + 8k^6 - 64k^5n + 34640k^2 - 1028112nk - 186251k) \cdot \\ & \frac{(-1/4)(8n + 3 - 2k)k}{(2n - k + 2)(2n - k + 1)(16n + 17)(16n + 15)(16n + 7)(16n + 9)(4n + 5)(4n + 3)} \quad \square \end{aligned}$$

Theorem 50 :

$$F(-2n, 4n + 1, -2n + 2, 1/3) = (81/16)^n \cdot \frac{(1/2)_n}{(-1/2)_n}.$$

Proof :

$$(1040n^3 + 1812n^2 + 264kn^2 + 982n + 36k^2n + 342kn + 24k^2 + 162 + 103k + 2k^3) \cdot \frac{(-1/54)(2n - k - 1)k}{n(2n - k + 2)(4n + 3)(2n + 1)(4n + 1)} \quad \square$$

Theorem 51 : Let $\zeta_2 = 1 \pm \sqrt{3}/3$, then

$$F(-2n, 4n + 2, 4/3, (2/3)\zeta_2) = (-9 + 18\zeta_2)^n \cdot \frac{(5/12)_n}{(5/4)_n}.$$

Proof :

$$(352n^2 + 432n + 168\zeta_2n + 116 + 138\zeta_2 - 2k^2 + 9\zeta_2k^2 - 288\zeta_2nk + 152nk + 126k - 237\zeta_2k) \cdot \frac{(-1/88)(-16 + 9\zeta_2)(1 + 3k)k}{(2n - k + 2)(2n - k + 1)(12n + 5)(4n + 3)} \quad \square$$

Theorem 52 : Let $\zeta_2 = 1 \pm \sqrt{3}/3$, then

$$F(-2n, 4n + 1, 2/3, (2/3)\zeta_2) = (-9 + 18\zeta_2)^n \cdot \frac{(7/12)_n}{(3/4)_n}.$$

Proof :

$$(352n^2 + 264n + 264\zeta_2n + 20 + 174\zeta_2 - 2k^2 + 9\zeta_2k^2 - 288\zeta_2nk + 152nk + 102k - 195\zeta_2k) \cdot \frac{(-1/88)(-16 + 9\zeta_2)(-1 + 3k)k}{(2n - k + 2)(2n - k + 1)(12n + 7)(4n + 1)} \quad \square$$