In how many ways can you play Stanley Solitaire?

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Dedicated to Algebraic Combinatorics guru, Richard Stanley (b. June 23, 1944) on his forthcoming 80th birthday

Preface: Stanley Solitaire: A Game You Can Teach a Five-Year-Old

It is noneducational to give candy to children unless they learn something from it. Here is a simple educational game that teaches children the notions of *larger* and *smaller*, and *left* and *right*. We named this game *Stanley Solitaire* for a reason to be explained later.

You start with an *initial position* on a one-dimensional **board**, where some squares have piles of candies (for example M&Ms) on them. The left side of the board is fixed, but one can move as far right as one wishes, following the rules.

One such starting position (out of infinitely many) could be

$$[4, 5, 0, 0, 2, 0, 3, 1] \quad ,$$

meaning that the rightmost location has 4 candies, the second-from-the-right has 5 candies, the third, fourth, and sixth locations are empty, the fifth has 2 candies, the seventh has 3 candies, and the eighth has one candy.

The player is allowed to eat all the candies, provided that he or she follow one strict rule.

Whenever in two adjacent locations the number of candies in the left location is (strictly) larger than the right location, exchange them, and eat one of the candies of the larger pile (the one that initially was on the right, and now is to the left).

If the player messes up, they are not allowed to play anymore, so they have a strong motivation to follow the rules.

In the above example, the legal moves are

• Eat one candy from the second location, and exchange the second and third, getting to the new position $[4, 0, 4, 0, 2, 0, 3, 1]$.

• Eat one candy from the fifth location, and exchange the fifth and the sixth piles, getting to the new position $[4, 5, 0, 0, 0, 1, 3, 1]$.

• Eat one candy from the seventh location, and exchange the seventh and eighth piles, getting to the new position $[4, 5, 0, 0, 2, 0, 1, 2]$.

• Eat one candy from the eighth location, and exchange the eighth and the (implied) ninth pile getting to the new position $[4, 5, 0, 0, 2, 0, 3, 0, 0]$, but we delete all rightmost zeroes, so the new position is $[4, 5, 0, 0, 2, 0, 3]$.

More formally, for a position

$$[a_1, \ldots, a_k] \quad ,$$

where we insist that $a_1 > 0$ and $a_k > 0$, and otherwise $a_i \geq 0$, the legal moves are For all $1 \leq i < k$, if $a_i > a_{i+1}$ then go to position

$$[a_1, \ldots, a_{i+1}, a_i - 1, \ldots, a_k] \quad ,$$

and another legal move is to position $[a_1, \ldots, a_{k-1}, 0, a_k - 1]$. Whenever the new position has 0 either on the left or right, we remove them, keeping the convention that the leftmost and rightmost piles are non-empty.

A problem you can Explain To a Five-Year-Old

If you start at a given position, say $[5, 4, 3, 2, 1]$, so you can eat 15 candies as long as you follow the rules, how many ways can you do it?

It turns out (see later) that the number is 292864.

If the initial position is $[2, 1]$, then there are only two ways:

$$[2, 1] \to [1, 1] \to [1] \to [] \quad ,$$

$$[2, 1] \to [2, 0, 0](\, alias\, [2]\,) \to [1] \to [] \quad .$$

While if the starting position is $[2, 2, 1]$ there are five ways:

$$[[2, 2, 1], [2, 1, 1], [1, 1, 1], [1, 1], [1], []] \quad ,$$

$$[[2, 2, 1], [2, 1, 1], [2, 1], [1, 1], [1], []] \quad ,$$

$$[[2, 2, 1], [2, 1, 1], [2, 1], [2], [1], []] \quad ,$$

$$[[2, 2, 1], [2, 2], [2, 0, 1], [1, 1], [1], []] \quad ,$$

$$[[2, 2, 1], [2, 2], [2, 0, 1], [2], [1], []] \quad .$$

An Answer you can Explain to a Seven-Year-Old

Thanks to the seminal work of Richard Stanley [S], and independently, Paul Edeleman, and Curtis Greene [EG] (and presumably [LSc], but it is over our heads), there is an extremely easy and elegant **answer**, that only involves addition and multiplication (recall that $a! = 1 \cdot 2 \cdots a$)

A Simply Stated but Surprisingly Hard-To-Prove Theorem ([S],[EG])

The **exact** number of ways to play Stanley-Solitaire starting with position,

$$[a_1, \ldots, a_k] \quad ,$$

where $a_1 \geq a_2 \geq \ldots \geq a_1 > 0$ is

$$\frac{(a_1 + \ldots + a_k)!}{(a_1+k-1)!(a_2+k-2)! \cdots (a_k)!} \cdot \prod_{1 \leq i < j \leq k} (a_i - a_j + j - i) \quad .$$

In fact, for any 213-avoiding permutation (there are $C_k = (2k)!/(k!(k+1)!)$ of them) the number of ways of Playing Stanley Solitaire with initial positions

$$[a_{\pi_1}, \ldots, a_{\pi_k}] \quad ,$$

is also given by that very same expression, namely

$$\frac{(a_1 + \ldots + a_k)!}{(a_1+k-1)!(a_2+k-2)! \cdots (a_k)!} \cdot \prod_{1 \leq j < i \leq k} (a_i - a_j + j - i) \quad .$$

If π is not 213-avoiding, then the formulas are no longer so nice (see below).

References

[EG] Paul Edeleman and Curtis Greene, *Balanced tableaux*, Adv. Math. **63** (1987), 42-99.
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[S] Richard Stanley, *On the number of reduced decompositions of elements of the Coxeter group*, Europ. J. Combinatorics **5** (1984), 359-372.
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