SOCCER GAME HISTORY

LECTURE GIVEN AT MATH LEAGE CAMP, JULY 23, 2018

BY DORON ZEILBERGER

JUNE 15, 2018

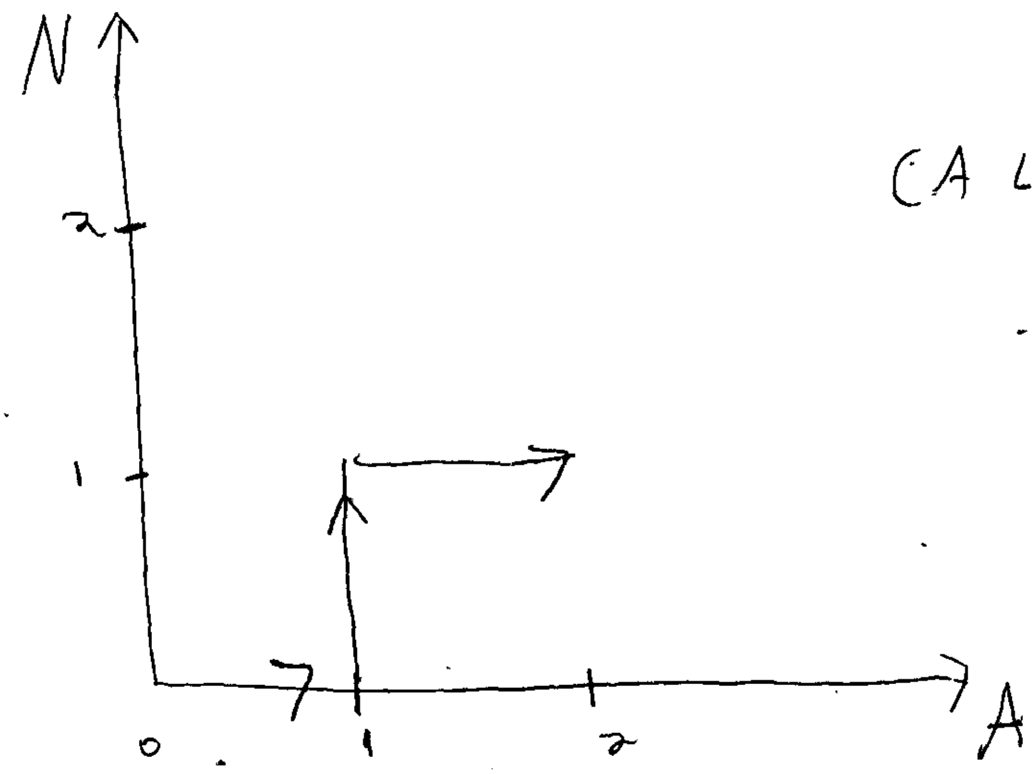
ARGENTINA: 2 | NIGERIA: 1

MIN	14	51	58
GOAL FOR	A	N	A

HISTORY: A N A

$(0,0) \to (1,0) \to (1,1) \longrightarrow (2,1)$

LATTICE WALK

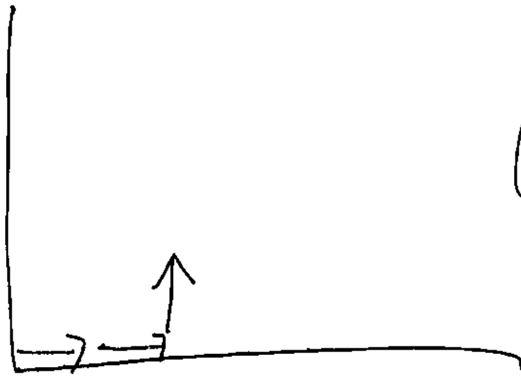


(A LITTLE BORING)

OTHER POSSIBLE HISTORIES

AAN

$(0,0) \to (1,0) \to (2,0) \to (2,1)$

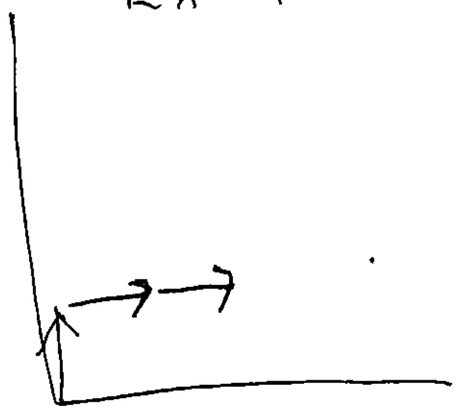


(VERY BORING)

NAA

EXCITING

$(0,0) \to (0,1) \to (1,1) \to (2,1)$



TOTAL NUMBER OF HISTORIES = 3

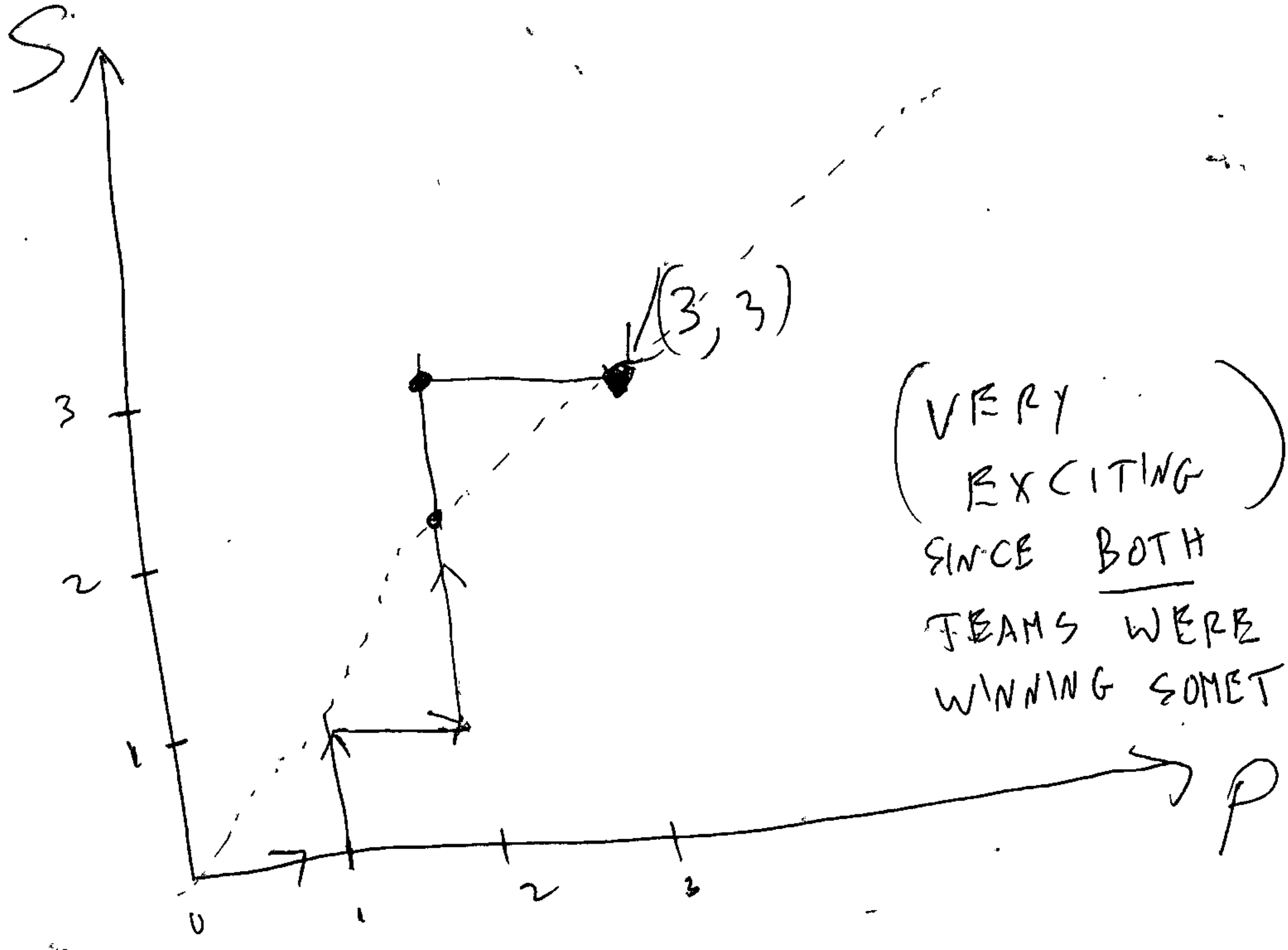
NUMBER OF VERY BORING HISTORIES = 1

JUNE 15, 2018

PORTUGAL: 3 SPAIN: 3

MIN	4	24	44	55	58	88
GOAL	P	S	P	S	S	P

(0,0) → (1,0) → (1,1) → (2,1) → (2,2) → (2,3) → (3,3)



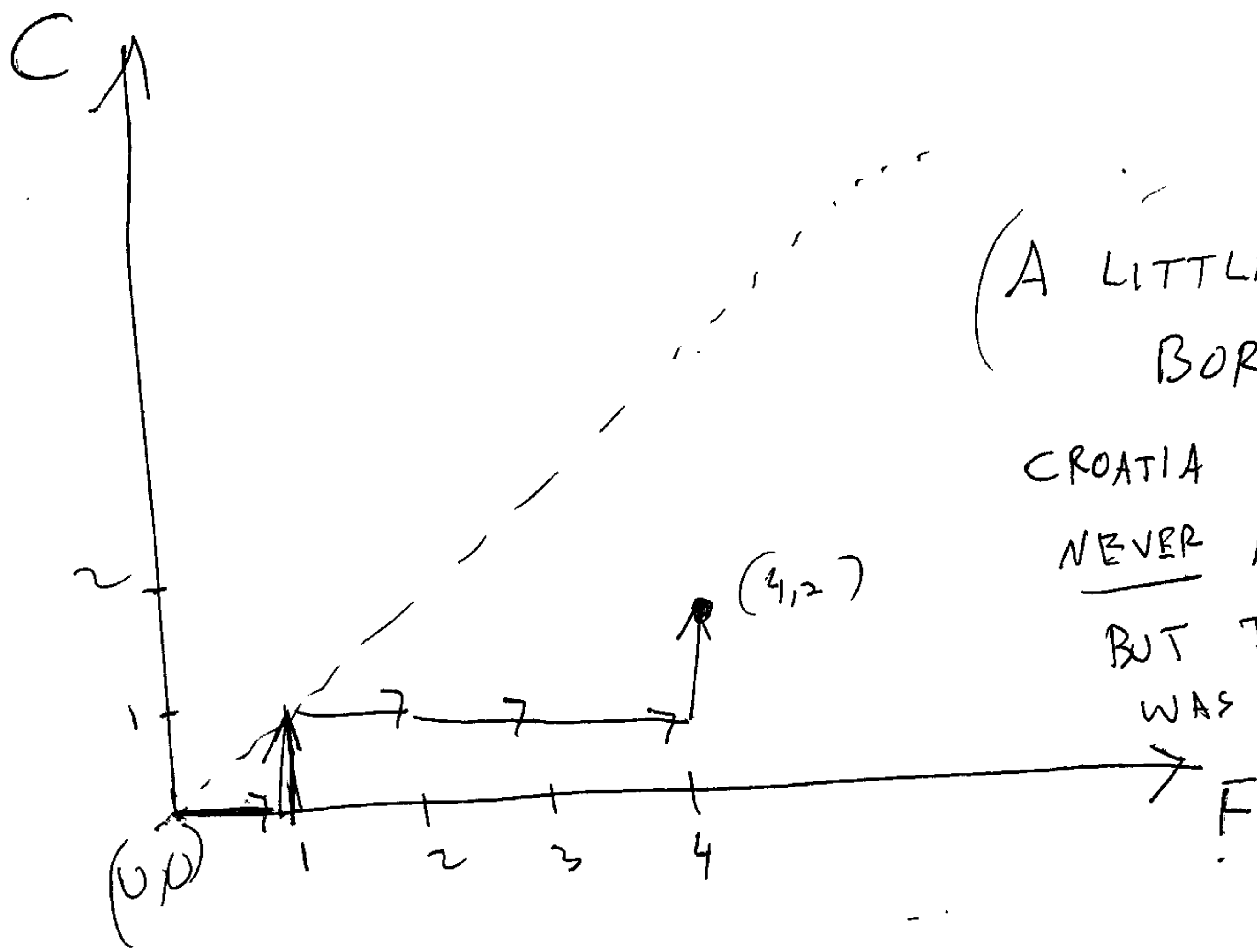
(VERY EXCITING) SINCE BOTH TEAMS WERE WINNING SOMETIM

FINAL : JULY 15, 2018

FRANCE: 4 CROATIA: 2

MW	18	28	38	59	65	69
GOAL	F	C	F	F	F	C

$(0,0) \to (1,0) \to (1,1) \to (2,1) \to (3,1) \to (4,1) \to (4,2)$



(A LITTLE BORING)

CROATIA WAS NEVER AHEAD BUT THERE WAS A TI

THEOREM:

NUMBER OF POSSIBLE HISTORIES IN A SOCCER GAME WITH FINAL SCORE A:B IS

$$\binom{A+B}{A} = \binom{A+B}{B} = \frac{(A+B)(A+B-1)\cdots(A+1)}{B\cdot(B-1)\cdots 1}$$

THEOREM:

NUMBER OF WALKS IN MANHATTAN FROM $(0,0)$ TO (A,B) IS

$$\frac{(A+B)(A+B-1)\cdots(A+1)}{B\cdot(B-1)\cdots 1} = \frac{(A+B)!}{A!\,B!}$$

EXAMPLE: NUMBER OF HISTORIES WITH 4:2

IS $\binom{6}{2} = \frac{6\cdot 5}{1\cdot 2} = 15$

FFFFBB ← VERY BORING
FFFBFB
FFFBBF
FFBFFB
.
.
.
.
.

15 possibilities

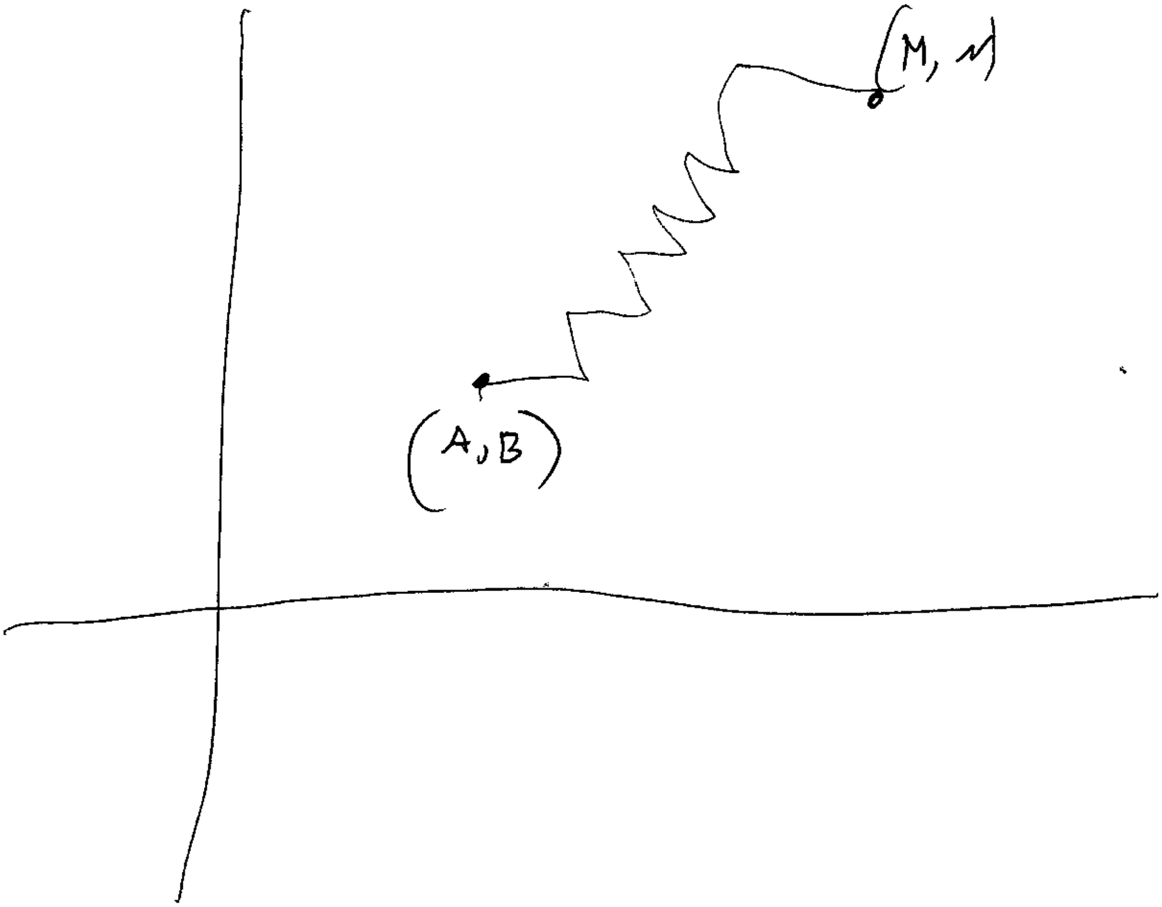
BBFFFF ← VERY EXCITING

FBBFFF ← EXCITING

FFBFFB ← VERY BORING

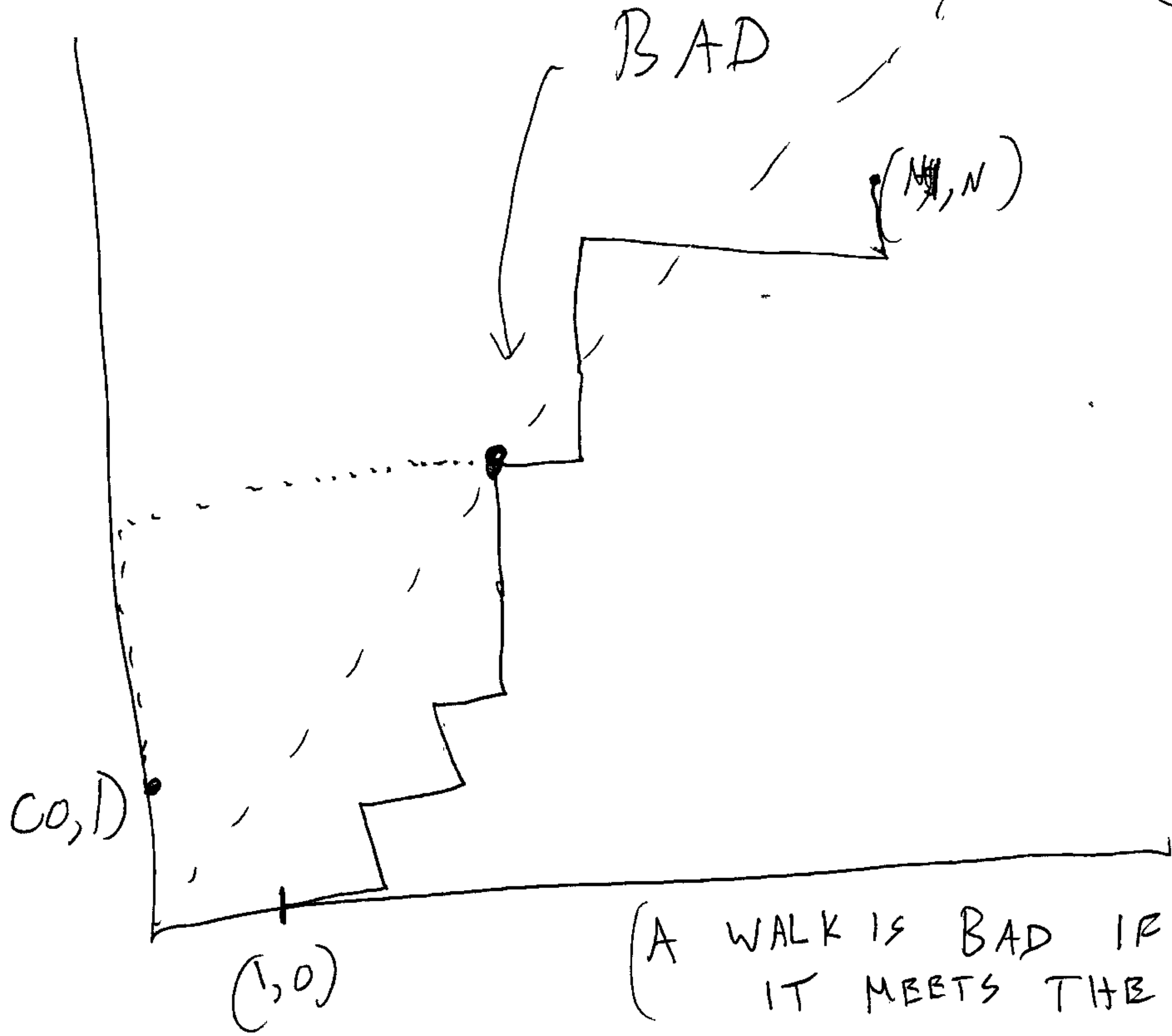
FFBBFF ← A LITTLE BORING

COR:



NUMBER OF WALKS FROM (A,B) to (M,N) IS

$$\binom{M-A+N-B}{M-A} = \binom{M-A+N-B}{N-B}$$



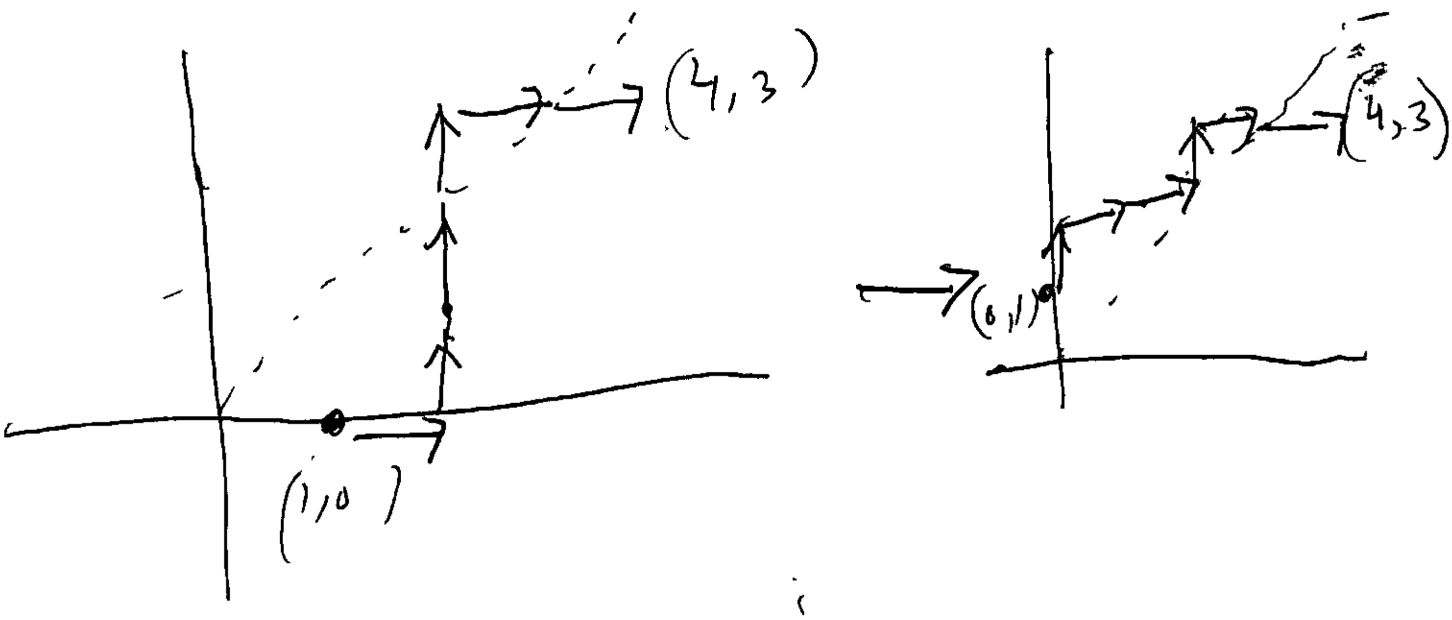
(A WALK IS BAD IF IT MEETS THE DIAGONAL)

\# OF BAD WALKS FROM (1,0) TO (M,N) =

\# OF BAD WALKS FROM (0,1) TO (M,N) = \# OF WALKS FROM (0,1) TO (M,N) $= \binom{M+N-1}{M}$

(SINCE ALL WALKS FROM (0,1) TO (M,N) ARE BAD)

REFLECTION PRINCIPLE



$(1,0) \to (2,0) \to (2,1) \to (2,2) \to (2,3) \to (3,3) \to (4,3)$

$(0,1) \to (0,2) \to (1,2) \to (2,2) \to (2,3) \to (3,3) \to (4,3)$

← REFLECT → ← KEEP →

ANDRÉ

FIRST TIME IT MEETS DIAGONAL

OF WALKS FROM
$(1,0)$ TO $(N+1,N) = \binom{2N}{N}$

OF BAD WALKS FROM
$(1,0)$ TO $(N+1,N) =$

OF WALKS FROM
$(0,1)$ TO $(N+1,N) = \binom{2N}{N-1}$

OF GOOD WALKS FROM
$(1,0)$ TO $(N+1,N) =$

$$\binom{2N}{N} - \binom{2N}{N-1} = \frac{1}{N+1}\binom{2N}{N}$$

CATALAN

CATALAN NUMBERS

$$\frac{1}{N+1}\binom{2N}{N} = \frac{1}{2N+1}\binom{2N+1}{N}$$

(VERY FAMOUS!)

ANOTHER PROOF

DEF: CYCLIC SHIFTS

AAABB	AAABB, AABBA, ABBAA, BBAAA, BAAAB
AABAB	AABAB, ABABA, BABAA, ABAAB, BAABA
GOOD WALKS	ALL WALKS

TOTAL NUMBER OF WALKS FROM $(0,0)$ TO $(N+1, N) =$

$$\binom{2N+1}{N}$$

EVERY GOOD WALK HAS $2N+1$ (DISTINCT!) CYCLIC SHIFTS

$\left[\#\text{ OF GOOD WALKS FROM } (0,0) \text{ TO } (N+1, N)\right] \cdot (2N+1) =$

$$\binom{2N+1}{N}$$

$\therefore$ # OF GOOD WALKS (VERY BORING GAMES)

$$= \frac{1}{2N+1}\binom{2N+1}{N} = \frac{1}{N+1}\binom{2N}{N}$$

HOW TO FIND THE (UNIQUE!) GOOD CYCLIC SHIFT?

-1 0 -1 -2 -1 0 -1 0 1

B A B B | A A B A A

LOWEST DIFFERENCE #A − #B

1 2 2 3 2 3 2 1

A A B A A B A B B

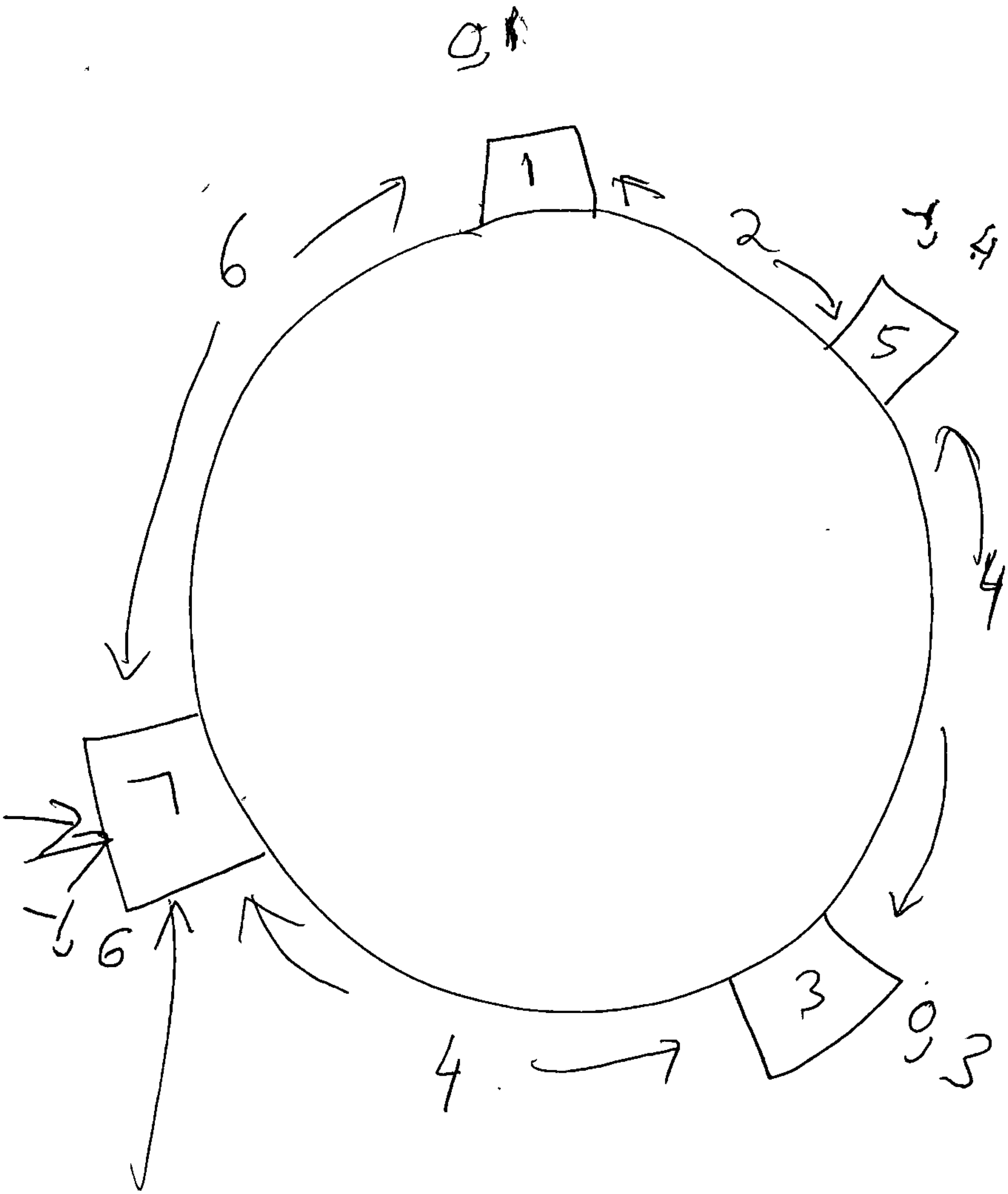
1 0 1 0 -1 0 -1 -2 -1 -2 | -1 0 1

A B A B B A B B A B | A A A

LAST TIME THE SMALLEST DIFFERENCE HAPPENED

1 2 3 4 3 4 3 2 3 2 1 2 1

A A A A B A B B A B B A B



START HERE NOT
TO RUN OUT OF GAS