IDENTITIES In CHARACTER TABLES Of S_n

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Abstract: *We use the algebra of difference operators to study sums of squares (and other powers) of the characters of the symmetric group, $\chi^\lambda(\mu)$, when the sum is restricted over shapes, λ, with a fixed number of rows, and for hook shapes, and μ has 'mostly' ones. We prove that such sums are always P-recursive, i.e., satisfy a linear difference equation with polynomial coefficients. For the special case of two rows, and for hook-shapes, we prove that these sums are in fact closed-form, and we present algorithms, complete with rigorous proofs, for finding these expressions.This article is accompanied by a Maple package,* `Sn` *(available from* `http://www.math.rutgers.edu/~zeilberg/tokhniot/Sn`*), and a webpage,* `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sn.html`*, with links to extensive output, containing rigorously-proved explicit formulas for many cases.*

Introduction

The notion of *group character*, introduced by Frobenius, is a fundamental one in representation theory, and among all groups, the symmetric group, S_n, is the most fundamental one. The entries of the character tables of the symmetric group may be defined in numerous ways, but for our purposes, we will use the following definition, that does not require any knowledge of 'advanced' algebra, high-school algebra suffices!

Recall that the *Constant Term* of a *Laurent polynomial* in $(x_1, \dots, x_m)$ is the free term, i.e. the coefficient of $x_1^0 \cdots x_m^0$. For example

$$CT_{x_1,x_2}(x_1^{-3}x_2 + x_1x_2^{-2} + 5) = 5 \quad .$$

Recall that a *partition* (alias *shape*) of an integer n, with m *parts* (alias *rows*), is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_m) \quad ,$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$, and $\lambda_1 + \dots + \lambda_m = n$.

If $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_r)$ are partitions of n with m and r parts, respectively, then it easily follows from (7.8) (p. 114) in [M], that the *characters*, $\chi^\lambda(\mu)$, of the *symmetric group*, S_n, may be obtained by the *constant term* expression

$$\chi^\lambda(\mu) \;=\; CT_{x_1,\dots,x_m} \frac{\prod_{1\leq i<j\leq m}(1-\frac{x_j}{x_i})\prod_{j=1}^{r}\left(\sum_{i=1}^{m} x_i^{\mu_j}\right)}{\prod_{i=1}^{m} x_i^{\lambda_i}} \quad . \qquad (Chi)$$

It is well-known (e.g. [M], p. 119, bottom line) that, writing μ in *frequency notation*, $\mu = 1^{a_1}2^{a_2}\dots n^{a_n}$, we have the following beautiful identity:

$$\sum_{\lambda\vdash n} \chi^\lambda(\mu)^2 \;=\; \prod_{i=1}^{n} i^{a_i} a_i! \quad .$$

The most famous special case is when $\mu = 1^n$, that becomes (since $\chi^\lambda(1^n) = f_\lambda$, the number of Standard Young Tableaux of shape λ), the identity:

$$\sum_{\lambda \vdash n} f_\lambda^2 \, = \, n! \quad ,$$

that has a lovely combinatorial proof using the celebrated Robinson-Schensted correspondence ([Ro][Sc]). A bijective proof of the former, more general, identity was given by Dennis White ([Wh]).

If one restricts the latter sum to go over partitions with at most a fixed number of parts, then one gets many sequences of combinatorial interest. Most notably, for a fixed $r \geq 1$,

$$a^{(r)}(n) \, := \, \sum_{\substack{\lambda \vdash n \\ length(\lambda) \leq r}} f_\lambda^2 \quad ,$$

is the number of permutations of length n that do not contain an increasing subsequence of length $r+1$. The sequences $a^{(r)}(n)$ are (as of July 8, 2015) in [Sl] for $r \leq 11$. Note that $a^{(2)}(n)$ is the super-famous sequence **A000108**, of Catalan numbers, $C_n := \frac{1}{n+1}\binom{2n}{n}$.

In [Z1] it was proved that, for any fixed r, the sequence $a^{(r)}(n)$ is P-recursive (alias *holonomic*), i.e. satisfies a *linear difference equation* (alias *linear recurrence equation*) with *polynomial coefficients*. We should also mention that Ira Gessel ([Ge]) famously discovered a lovely determinant formula for the generating functions in terms of Bessel functions.

The analogous sequences, for the straight sums (without the squares),

$$b^{(r)}(n) \, := \, \sum_{\substack{\lambda \vdash n \\ length(\lambda) \leq r}} f_\lambda \quad ,$$

are also of combinatorial interest, counting the number of *involutions* of length n avoiding an increasing subsequence of length $r+1$. It is well known, and easy to prove, that $b^{(2)}(n) = \binom{n}{\lfloor n/2 \rfloor}$, that is sequence **A001405** in [Sl]. Much deeper is the result, first proved in [Re], that $b^{(3)}(n)$ are the *Motzkin numbers*, **A001006** (see [Z2] for another proof and for a generalization). The sequence $b^{(4)}(n)$, **A005817**, is even nicer, given in terms of Catalan numbers, $b^{(4)}(n) = C_{\lfloor n/2+1/2 \rfloor} C_{\lfloor n/2+1 \rfloor}$, as first proved by Dominique Gouyou-Beauchamps ([Go]). $b^{(5)}(n)$ is **A049401**, while $b^{(6)}(n)$ is **A007579**. See also [BFK].

Recall ([Z1]) that a discrete function $a(n)$ of a *single* variable is called *holonomic* (or P-recursive) if it satisfies a homogeneous linear recurrence (alias *difference*) equation with polynomial coefficients, i.e. there exists an integer L and polynomials $p_0(n), \dots, p_L(n)$, such that

$$\sum_{i=0}^{L} p_i(n) a(n+i) \, = 0 \quad , \quad (n \geq 0) \quad .$$

A discrete function of several variables $a(n_1, \dots, n_m)$ is holonomic if it satisfies such a recurrence in each of its variables, $n_1, \dots, n_m$ and the coefficients are polynomials in all of them, and the system is non-degenerate. It was proved in [Z1] that if you sum such a holonomic discrete function over some of its arguments, you get yet-another holonomic function in the surviving variables.

As mentioned in [Z1], the reason that $a^{(r)}(n)$ and $b^{(r)}(n)$ (and more generally, sums of powers f_λ^s, where s is any positive integer), for any *specific*, r, are P-recursive in n is that the *summand*, $f_\lambda = f(\lambda_1, \dots, \lambda_r)$, and hence any of its powers, is *holonomic* in its arguments, and hence the $(r-1)$-fold *multisum* is guaranteed to be holonomic in the 'surviving' discrete variable n. Furthermore, thanks to the *hook-length* formula ([Wi]), or equivalently the *Young-Frobenius* formula, the summand $f(\lambda_1, \dots, \lambda_r)$ can be expressed as

$$f(\lambda_1, \dots, \lambda_r) \, = \, \frac{\prod_{1 \leq i < j \leq r} (\lambda_j - \lambda_i + j - i)(\sum_{i=1}^r \lambda_i)!}{\prod_{i=1}^r (\lambda_i + r - i)!} \quad .$$

In addition, thanks to [AZ], there are effective algorithms for finding these recurrences, implemented in the Maple package
`http://www.math.rutgers.edu/~zeilberg/tokhniot/MultiZeilberger` .

However, it is much more efficient to derive these recurrences by generating sufficiently many terms, and then *guessing* the linear recurrence, that we know for sure exists, by *undetermined coefficients*. These recurrences can be proved fully rigorously, if desired, but since they are definitely true, we do not bother to waste time on this.

More generally, one can consider such sums where the shape λ belongs to a "meta-hook" with k rows and l columns, in other words, the analogous sums (see [BR] and [EZ]) where one sums f_λ (or its square, or any positive integer power), over all shapes, that do **not** contain the celll $(k+1, l+1)$. Such a shape is determined by $k+l$ parameters, the lengths of the k largest rows and the lengths of the l largest columns, and, once again, using the Hook Length Formula, one can express f_λ as a closed-form expression in these $k+l$ discrete parameters, and the above observations about the resulting sequences, for each *fixed* (k, l), being holonomic, still apply.

Character Sums for μ 'Close' to 1^n

For any partition λ, let $|\lambda|$ be its sum, in other words, the integer that is being partitioned.

As we noted above, f_λ equals $\chi^\lambda(1^n)$. The first purpose of the present paper is merely to **observe**, that an analogous argument still applies if one replaces $\mu = 1^n$ by $\mu = \mu_0 1^{n-|\mu_0|}$, for *any* fixed partition μ_0 (with smallest part at least 2), the analogous sums with f_λ replaced by $\chi^\lambda(\mu_0 1^{n-|\mu_0|})$ are also *guaranteed* to be holonomic. This follows from Eq. (Chi), that spells out to be, writing $\mu_0 = (a_1, \dots, a_k)$ $(a_k \geq 2)$.

$$\chi^\lambda(\mu_0 1^{n-|\mu_0|}) \, = \, CT_{x_1, \dots, x_m} \, \frac{\prod_{1 \leq i < j \leq m} (1 - \frac{x_j}{x_i}) \prod_{j=1}^k \left(\sum_{i=1}^m x_i^{a_j} \right) \cdot \left(\sum_{i=1}^m x_i \right)^{n - a_1 - \dots - a_k}}{\prod_{i=1}^m x_i^{\lambda_i}} \quad . \qquad (Chi0)$$

Expanding

$$\prod_{1\le i<j\le m} (1-\frac{x_j}{x_i}) \prod_{j=1}^{k} \left(\sum_{i=1}^{m} x_i^{a_j} \right) \quad ,$$

we get, for *fixed* m, k, and $\mu_0 = (a_1, \dots, a_k)$, a **finite** sum of monomials, and collecting the contributions of each, we get a *finite* linear combination of shifts of the multinomial coefficient $(\lambda_1 + \ldots + \lambda_m)!/(\lambda_1! \cdots \lambda_m!)$, and it is easy to see that the result is a rational function times the latter (with 'nice' denominator), and hence closed-form.

The analogous argument for sums over shapes contained in a meta-hook $H(k,l)$ is slightly more complicated, and is omitted.

The second purpose of the present paper is to observe that, for the special cases of shapes with at most 2 rows, defining

$$\psi^{(2)}(\mu) := \sum_{j=0}^{\lfloor n/2 \rfloor} \chi^{(n-j,j)}(\mu)^2 \quad ,$$

for *any* fixed partition μ_0 (with smallest part ≥ 2), there is a closed-form expression for $\psi^{(2)}(\mu_0 1^{n-|\mu_0|})$, of the form

$$\psi^{(2)}(\mu_0 1^{n-|\mu_0|}) = R_{\mu_0}(n) \binom{2n}{n} \quad ,$$

for some **rational function** $R_{\mu_0}(n)$. The reason is that in the two-rowed case, $\chi^{(n-j,j)}(\mu_0 1^{n-\mu_0})$ can be expressed (thanks to Eq. (Chi)) as a linear combination of *shifts* of the binomial coefficients $\binom{n}{j}$, and squaring it and expanding, gives (possibly many, but still finitely-many) expressions of the form

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-\alpha}{j-\beta} \binom{n-\alpha'}{j-\beta'} \quad , \quad (\alpha+\alpha' = |\mu_0|) \quad ,$$

each of which (after symmetrizing in order to make the summations range over all $-\infty < j < \infty$ [with the usual convention that $\binom{a}{b}$ is 0 if $a<b$ and if $b<0$]) is summable by the Vandermonde-Chu convolution

$$\sum_j \binom{n-\alpha}{j-\beta} \binom{n-\alpha'}{j-\beta'} = \binom{2n-\alpha-\alpha'}{n+\beta'-\alpha'-\beta},$$

each of which is a multiple of $\binom{2n}{n}$ by a certain rational function, and adding these finitely (but possibly numerous) terms, still adds up to a certain rational function times $\binom{2n}{n}$.

Analogously, for shapes inside the $(1,1)$-meta hook

$$\phi^{(2)}(\mu) := \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu)^2 \quad ,$$

since $\chi^{(j,1^{n-j})}(1^n) = \binom{n-1}{j-1}$ (as follows easily from (Chi) specialized to this case), $\chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})$ can be expressed as a finite linear combinations of $\binom{n-1-\alpha}{j-1-\beta}$, once again we get a finite linear combination of Vandermonde-Chu convolutions, each of them being a multiple of $\binom{2n-2}{n-1}$ by a certain rational function (equivalently, we can use $\binom{2n}{n}$, as above, but it is more natural to use the former as the "base", since it is the answer for $\mu = 1^n$, i.e. where μ_0 is the *empty* partition).

In fact, in this case we can get nice explicit expressions for the generation functions (recall that $\mu_0 = (a_1, \dots, a_r)$)

$$F_{n;(a_1,\dots,a_r)}(x) := \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})\, x^j \;=\; x(1+x)^{n-1-|\mu_0|} \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i}) \quad .$$

Hence

$$\phi^{(2)}(\mu_0 1^{n-|\mu_0|}) \;=\; \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})^2$$

is the *constant term* of

$$F_{n;(a_1,\dots,a_r)}(x) \cdot F_{n;(a_1,\dots,a_r)}(x^{-1}) =$$

$$x(1+x)^{n-1-a_1-\dots-a_r} \left(\prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i}) \right) \cdot x^{-1}(1+x^{-1})^{n-1-a_1-\dots-a_r} \left(\prod_{i=1}^{r} (x^{-a_i} - (-1)^{a_i}) \right) \quad .$$

$$= \frac{(1+x)^{2n-2-2|\mu_0|}}{x^{n-1-|\mu_0|}} \cdot Q(x) \quad ,$$

where $Q(x)$ is the symmetric Laurent polynomial

$$Q(x) = \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i})(x^{-a_i} - (-1)^{a_i}) \quad .$$

Expanding $Q(x)$ as a sum of monomials and extracting the respective coefficients, we get a linear combination of terms of the form $\binom{2(n-1-|\mu_0|)}{n-1-|\mu_0|-j}$, that obviously simplifies to a rational function times $\binom{2n-2}{n-1}$. [This is implemented in procedure `Phi2` in the Maple package `Sn`].

Implementation

Everything discussed here is implemented in the Maple package `Sn`, available from the front of the present paper

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sn.html` ,

where one can find several sample input and output files, that readers are welcome to extend.

Some Output

I. **Explicit Expressions for $\boldsymbol{\phi_n^{(2)}(\mu_0 1^{n-|\mu_0|})}$**

The classical case is well-known, and easy.

$$\phi_n^{(2)}(1^n) = \binom{2n-2}{n-1} \quad .$$

We also have

$$\phi_n^{(2)}(21^{n-2}) = \frac{1}{2n-3}\binom{2n-2}{n-1} \quad (= 2C_{n-2}) \quad ,$$

$$\phi_n^{(2)}(221^{n-4}) = \frac{3}{(2\,n-3)\,(2\,n-5)}\binom{2n-2}{n-1} \quad .$$

More generally, for $r \geq 0$, we have:

$$\phi_n^{(2)}(2^r1^{n-2r}) = \frac{(2r)!(2n-2r-2)!}{r!(n-1)!(n-r-1)!} \quad .$$

Also

$$\phi_n^{(2)}(31^{n-3}) = \frac{n^2\ -7\,n\ +18}{4\,(2\,n-3)\,(2\,n-5)}\binom{2n-2}{n-1} \quad ,$$

$$\phi_n^{(2)}(41^{n-4}) = \frac{n^2-9\,n+23}{(2\,n-3)\,(2\,n-5)\,(2\,n-7)}\binom{2n-2}{n-1} \quad ,$$

$$\phi_n^{(2)}(51^{n-5}) = \frac{n^4-22\,n^3+239\,n^2-1298n+2760}{16\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)\,(2\,n-9)}\binom{2n-2}{n-1} \quad ,$$

$$\phi_n^{(2)}(321^{n-5}) = \frac{n^2-15\,n+74}{4\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)}\binom{2n-2}{n-1} \quad .$$

For *all* the (proved!) explicit expressions for $\phi_n^{(2)}(\mu_0 1^{n-|\mu_0|})$, with $|\mu_0| \leq 14$ (and, of course, the smallest part of μ_0 larger than one) (135 cases altogether), see the output file:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oSn1` .

II. **Explicit Expressions for $\psi_{\mathbf{n}}^{(\mathbf{2})}(\mu_{\mathbf{0}}\mathbf{1}^{\mathbf{n}-|\mu_{\mathbf{0}}|})$**

The classical case is well-known, and easy.

$$\psi_n^{(2)}(1^n) = \frac{1}{n+1}\binom{2n}{n} \quad (= C_n) \quad .$$

We have:

$$\psi_n^{(2)}(21^{n-2}) = \frac{9-5\,n+n^2}{(2\,n-1)\,(2\,n-3)\,(n+1)}\binom{2n}{n} \quad ,$$

$$\psi_n^{(2)}(31^{n-3}) = \frac{48-11\,n+n^2}{4\,(2\,n-1)\,(2\,n-3)\,(n+1)}\binom{2n}{n} \quad ,$$

Note the **remarkable** (proved!) identity:

$$\psi_n^{(2)}(31^{n-3}) = \frac{1}{2}\phi_{n+2}^{(2)}(321^{n-3}) \quad .$$

It may be interesting to find a 'natural' reason for this 'coincidence'.

We also have:

$$\psi_n^{(2)}(41^{n-4}) = \frac{2100 - 1354\,n + 299\,n^2 - 26\,n^3 + n^4}{4\,(2\,n-1)\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)\,(n+1)} \binom{2n}{n} \quad ,$$

$$\psi_n^{(2)}(221^{n-4}) = \frac{525 - 316\,n + 89\,n^2 - 14\,n^3 + n^4}{(2\,n-1)\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)\,(n+1)} \binom{2n}{n} \quad ,$$

$$\psi_n^{(2)}(51^{n-5}) = \frac{10080 - 4342\,n + 659\,n^2 - 38\,n^3 + n^4}{16\,\,(2\,n-1)\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)\,(n+1)} \binom{2n}{n} \quad ,$$

$$\psi_n^{(2)}(321^{n-5}) = \frac{2520 - 1045\,n + 194\,n^2 - 20\,n^3 + n^4}{4\,\,(2\,n-1)\,(2\,n-3)\,(2\,n-5)\,(2\,n-7)\,(n+1)} \binom{2n}{n} \quad .$$

For *all* the (proved!) explicit expressions for $\psi_n^{(2)}(\mu_0 1^{n-|\mu_0|})$, with $|\mu_0| \leq 14$ (and, of course, the smallest part of μ_0 larger than one) (135 cases altogether), see the output file:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oSn2` .

Sums over shapes with more rows

For three and more rows, the sums are no longer closed-form, but, as we mentioned above, they always satisfy a linear recurrence (alias difference) equation with polynomial coefficients, see the output files linked to in the above-mentioned webpage of this article.

Conclusion

On page 155 of [GPK] it says:

"The numbers in Pascal's triangle satisfy, practically speaking, infinitely many identities, so it is not too surprising that we can find some surprising relationships by looking closely."

The aim of this note was to indicate that a similar statement seems to hold for the character tables of the symmetric groups Sn. Just as importantly, it was a *case-study* in using a computer algebra system to prove deep identities, way beyond the ability of mere humans.

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