Abstract: In a recent article, we noted (and proved) that the sum of the squares of the characters of the symmetric group, \( \chi^{\lambda}(\mu) \), over all shapes \( \lambda \) with two rows and \( n \) cells and \( \mu = 31^{n-3} \), equals, surprisingly, to \( 1/2 \) of that sum-of-squares taken over all hook shapes with \( n+2 \) cells and with \( \mu = 321^{n-3} \). In the present note, we show that this is only the tip of a huge iceberg! We will prove that if \( \mu \) consists of odd parts and (a possibly empty) string of consecutive powers of 2, namely \( 2, 4, \ldots, 2^{t-1} \) for \( t \geq 1 \), then the the sum of \( \chi^{\lambda}(\mu)^2 \) over all two-rowed shapes \( \lambda \) with \( n \) cells, equals exactly \( 1/2 \) times the analogous sum of \( \chi^{\lambda}(\mu')^2 \) over all shapes \( \lambda \) of hook shape with \( n+2 \) cells, and where \( \mu' \) is the partition obtained from \( \mu \) by retaining all odd parts, but replacing the string \( 2, 4, \ldots, 2^{t-1} \) by \( 2^t \).

Recall that the Constant Term of a Laurent polynomial in \((x_1, \ldots, x_m)\) is the free term, i.e. the coefficient of \( x_1^0 \cdots x_m^0 \). For example

\[
CT_{x_1, x_2}(x_1^{-3} x_2 + x_1 x_2^{-2} + 5) = 5 .
\]

Recall that a partition (alias shape) of an integer \( n \), with \( m \) parts (alias rows), is a non-increasing sequence of positive integers

\[
\lambda = (\lambda_1, \ldots, \lambda_m) ,
\]

where \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0 \), and \( \lambda_1 + \ldots + \lambda_m = n \).

If \( \lambda = (\lambda_1, \ldots, \lambda_m) \) and \( \mu = (\mu_1, \ldots, \mu_r) \) are partitions of \( n \) with \( m \) and \( r \) parts, respectively, then it easily follows from (7.8) (p. 114) in [M], that the characters, \( \chi^{\lambda}(\mu) \), of the symmetric group, \( S_n \), may be obtained via the constant term expression

\[
\chi^{\lambda}(\mu) = CT_{x_1, \ldots, x_m} \left( \prod_{1 \leq i < j \leq m}(1 - \frac{x_j}{x_i}) \prod_{j=1}^{m} \left( \sum_{i=1}^{m} x_i^{\mu_j} \right) \right) .
\]  

(Chi)

As usual, for any partition \( \mu \), \( |\mu| \) denotes the sum of its parts, in other words, the integer that is being partitioned.

In [RRZ] we considered two quantities. Let \( \mu_0 \) be any partition with smallest part \( \geq 2 \). The first quantity, that we will call henceforth \( A(\mu_0)(n) \), is the following sum-of-squares over two-rowed shapes \( \lambda \):

\[
A(\mu_0)(n) := \sum_{j=0}^{[n/2]} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 .
\]

[Note that in [RRZ] this quantity was denoted by \( \psi^{(2)}(\mu_0 1^{n-|\mu_0|}) \).]
The second quantity was the sum-of-squares over hook-shapes

\[ B(\mu_0)(n) := \sum_{j=1}^{n} \chi(j,1^{n-j})(\mu_0 1^{n-|\mu_0|})^2 \, . \]

[Note that in RRZ this quantity was denoted by \( \phi^{(2)}(\mu_0 1^{n-|\mu_0|}) \).]

In RRZ we developed algorithms for discovering (and then proving) closed-form expressions for these quantities, for any given (specific) finite partition \( \mu_0 \) with smallest part larger than one. In fact we proved that each such expression is always a multiple of \( \binom{2n}{n} \) by a certain rational function of \( n \) that depends on \( \mu_0 \).

Unless \( \mu_0 \) is very small, these rational functions turn out to be very complicated, but, inspired by the OEIS([S]), Alon Regev noted (and then it was proved in RRZ) the remarkable identity

\[ A(3)(n) = \frac{1}{2} B(3, 2)(n + 2) \, . \]

This lead to the following natural question:

Are there other partitions, \( \mu_0 \), such that there exists a partition, \( \mu'_0 \) with \( |\mu'_0| = |\mu_0| + 2 \), such that the ratio \( A(\mu_0)(n)/B(\mu'_0)(n + 2) \) is a constant?

This lead us to write a new procedure in the Maple package

http://www.math.rutgers.edu/~zeilberg/tokhniot/Sn.txt , that accompanies RRZ,

called SeferNisim(K,N0), that searched for such pairs \([\mu_0, \mu'_0]\). We then used our human ability for pattern recognition to notice that all the successful pairs (we went up to \( |\mu_0| \leq 20 \)) turned out to be such that \( \mu_0 \) either consisted of only odd parts, and then \( \mu'_0 \) was \( \mu_0 \) with 2 appended, or, more generally \( \mu_0 \) consisted of odd parts together with a string of consecutive powers of 2 (starting with 2), and \( \mu'_0 \) was obtained from \( \mu_0 \) by retaining all the odd parts but replacing the string of powers of 2 by a single power of 2, one higher then the highest in \( \mu_0 \). In symbols, we conjectured, (and later proved [see below], alas, by purely human means) the following:

**Theorem:** Let \( \mu_0 \) be any partition of the form

\[ \mu_0 = \text{Sort}([a_1, \ldots, a_s, 2, 2^2, \ldots, 2^{t-1}]) \, , \]

where

\[ a_1 \geq a_2 \geq \ldots \geq a_s \geq 3 \, , \]

are all odd, and \( t \geq 1 \) (if \( t = 1 \) then \( \mu_0 \) only consists of odd parts). Define

\[ \mu'_0 = \text{Sort}([a_1, \ldots, a_s, 2^t]) \, . \]
Then, for every \( n \geq |\mu_0| \), we have

\[
A(\mu_0)(n) = \frac{1}{2} B(\mu'_0)(n + 2) .
\]

(For any sequence of integers, \( S \), \( \text{Sort}(S) \) denotes that sequence sorted in non-increasing order.)

In order to prove our theorem we need to first recall, from [RRZ], the following constant-term expression for \( B(\mu_0)(n) \).

**Lemma 1:** Let \( \mu_0 = (a_1, \ldots, a_r) \)

\[
B(\mu_0)(n) = \text{Coeff}_{x^{n-1}} \left[ \frac{(1 + x)^{2n-2 - 2(a_1 + \cdots + a_r)}}{x^{n-1}} \cdot \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i})(1 - (-1)^{a_i} x^{a_i}) \right] .
\]

We need an analogous constant-term expression for \( A(\mu_0)(n) \). To that end, let’s first spell-out Equation (\( \text{Chi} \)) for the two-rowed case, \( m = 2 \), so that we can write \( \lambda = (n - j, j) \). We have, writing \( \mu_0 = (a_1, \ldots, a_r) \),

\[
\chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) = CT_{x_1, x_2} \frac{(1 - \frac{a_2}{x_1})(x_1 + x_2)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (x_1^{a_i} + x_2^{a_i})}{x_1^{n-j} x_2^j} . \tag{\( \text{Chi}2 \)}
\]

This can be rewritten as

\[
\chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) = CT_{x_1, x_2} \frac{(1 - \frac{a_2}{x_1})(1 + \frac{a_2}{x_1})^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (1 + (\frac{a_2}{x_1})^{a_i})}{(\frac{a_2}{x_1})^j} . \tag{\( \text{Chi}2' \)}
\]

Since the constant-term and is of the form \( P(\frac{a_2}{x_1})/(\frac{a_2}{x_1})^j \) for some single-variable polynomial \( P(x) \), the above can be rewritten, as

\[
\chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) = \text{Coeff}_{x^0} \left[ (1 - x)(1 + x)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (1 + x^{a_i}) \right] . \tag{\( \text{Chi}2'' \)}
\]

Note that the left side is utter nonsense if \( j > \frac{n}{2} \), but the right side makes perfect sense. It is easy to see that defining \( \chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) \) by the right side for \( j > \frac{n}{2} \), we get

\[
\chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) = -\chi^{(j,n-j)}(\mu_01^{n-|\mu_0|}) .
\]

Let’s denote the numerator of the constant-term of (\( \text{Chi}'' \)), namely

\[
(1 - x)(1 + x)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (1 + x^{a_i}) ,
\]

by \( P(x) \), then equation (\( \text{Chi}2'' \)) can be also rewritten as a generating function.

\[
P(x) = \sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_01^{n-|\mu_0|}) x^j .
\]
Since for any polynomial of a single variable, $P(x) = \sum_{j=0}^{n} c_j x^j$, we have
\[
\sum_{j=0}^{n} c_j^2 = \text{Coeff}_{x^0} [P(x) P(x^{-1})],
\]
we get
\[
\sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 =
\]
\[
\text{Coeff}_{x^0} \left[ \left( (1-x)(1+x)^{n-a_1-\cdots-a_r} \prod_{j=1}^{r} (1+x^{a_j}) \right) \cdot \left( (1-x^{-1})(1+x^{-1})^{n-a_1-\cdots-a_r} \prod_{j=1}^{r} (1+x^{-a_j}) \right) \right].
\]
\[
= -\text{Coeff}_{x^0} \left[ (1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)} \prod_{j=1}^{r} (1+x^{a_j})^2 \right].
\]
But since, by symmetry,
\[
\sum_{j=0}^{\lfloor n/2 \rfloor} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 = \frac{1}{2} \sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2,
\]
we have

**Lemma 2:** Let $\mu_0 = (a_1, \ldots, a_r)$ be a partition with smallest part larger than one, then
\[
A(\mu_0)(n) = \frac{1}{2} \text{Coeff}_{x^0} \left[ (1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)} \prod_{j=1}^{r} (1+x^{a_j})^2 \right].
\]

We are now ready to prove the theorem. If $\mu_0 = \text{Sort}(a_1, \ldots, a_r, 2, \ldots, 2^{t-1})$ then
\[
A(\mu_0)(n) = \frac{1}{2} \text{Coeff}_{x^0} \left[ \frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-2^2-\cdots-2^{t-1})} \prod_{j=1}^{r-t} (1+x^{2^j})^2 \prod_{j=0}^{t-1} (1+x^{2^j})^2}{x^{n+1}} \right].
\]
But (transferring a factor of $(1+x)^2$ from the second factor to the product, $\prod_{j=1}^{r-t} (1+x^{2^j})^2$), we have
\[
(1+x)^{2(n-a_1-\cdots-a_r-2^2-\cdots-2^{t-1})} \prod_{j=1}^{r-t} (1+x^{2^j})^2 = (1+x)^{2(n-a_1-\cdots-a_r-1-2^2-\cdots-2^{t-1})} \prod_{j=0}^{t-1} (1+x^{2^j})^2.
\]
Hence,
\[
A(\mu_0)(n) = \frac{1}{2} \text{Coeff}_{x^0} \left[ \frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-1-2^2-\cdots-2^{t-1})} \prod_{j=0}^{t-1} (1+x^{2^j})^2 \prod_{j=1}^{r} (1+x^{a_j})^2}{x^{n+1}} \right].
\]
By Euler’s good-old \((1 - x) \prod_{j=0}^{t-1}(1 + x^{2^j}) = 1 - x^{2^t}\). Hence

\[
A(\mu_0)(n) = -\frac{1}{2} \text{Coeff}_{x^0} \left[ \frac{(1 - x^{2^t})^2(1 + x)^{2(n-a_1-\ldots-a_r-1-2^2-\ldots-2^{t-1})} \prod_{j=1}^r (1 + x^{a_j})^2}{x^{n+1}} \right].
\]

On the other hand, since \(\mu_0' = \text{Sort}(a_1, \ldots, a_r, 2^t)\), and all the \(a_i\)'s are odd, we have

\[
B(\mu_0')(n + 2) = -\text{Coeff}_{x^0} \left[ \frac{(1 + x)^{2n+2-2(a_1+\ldots+a_r+2^t)} x^{n+1}}{(x^{2^t} - 1)^2} \cdot (x^{2^t} - 1) \prod_{j=1}^r (x^{a_j} + 1)^2 \right].
\]

This completes the proof, since \(-(1 + 2 + 2^2 + \ldots + 2^{t-1}) = 1 - 2^t\). \(\square\)

Acknowledgment

The research for this work was done while the second-named author visited the Faculty of Mathematics at the Weizmann Institute of Science, during the week of Oct. 5-9, 2015. He wishes to thank the Weizmann Institute for its hospitality, and its dedicated stuff, most notably Gizel Maimon.

References


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